A Deformation of the Class Number Formula of Real Quadratic Fields

By S. HAHN and H. J. KIM

Korea Advanced Institute of Science and Technology

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1995)

Abstract: For an odd square-free integer n, there exists a polynomial $L_n(x)$ such that

$$L_n(x) = \sqrt{\phi_n(sx^2)}\exp(-s'\sqrt{n}g_n(x))$$

where $g_n(x) = \sum_{j=0}^{\infty}\left(\frac{n}{2j+1}\right)\frac{x^{2j+1}}{2j+1}$ and $s, s' = \pm 1$.

Using the fact that the value of $g_n(1)$ is related to the class number $h(D)$ of the real quadratic field $\boldsymbol{Q}(\sqrt{n})$ with discriminant D, we deduce a deformation of the class number formula.

First, we set the following notation.

$\phi(n)$ is the Euler function.

$\left(\frac{d}{n}\right)$ is the Jacobi symbol.

$\mu(n)$ is the Möbius function.

$\Phi_n(x)$ is the n-th cyclotomic polynomial.

ζ_n is a primitive n-th root of unity.

(n, k) is the greatest common divisor of k and n.

Let n be a positive odd square-free integer. For given n, we define integers n', s, and s' as follows:

$$n' = \begin{cases} n, & \text{if } n \equiv 1 \bmod 4, \\ 2n, & \text{otherwise.} \end{cases}$$

$$w = \begin{cases} 0, & \text{if } n \text{ has at least two distinct prime factors,} \\ 1, & \text{otherwise.} \end{cases}$$

$$s = \begin{cases} -1, & \text{if } n \equiv 3 \bmod 4, \\ 1, & \text{otherwise.} \end{cases}$$

$$s' = \begin{cases} -1, & \text{if } n \equiv 5 \bmod 8, \\ 1, & \text{otherwise.} \end{cases}$$

If we choose ζ_n to be $e^{\frac{2\pi i}{n}}$, then

$$(1)\quad 2G_n(x) := 2\prod_{\substack{0<j<n \\ (\frac{j}{n})=1}}(x-\zeta_n^j) = A_n(x) - \sqrt{sn}B_n(x),$$

$$(2)\quad 2\tilde{G}_n(x) := 2\prod_{\substack{0<j<n \\ (\frac{j}{n})=-1}}(x-\zeta_n^j) = A_n(x) + \sqrt{sn}B_n(x),$$

where $A_n(x), B_n(x) \in \boldsymbol{Z}[x]$. Note that the particular choice of ζ_n is only significant for the sign of $\sqrt{sn}$ and that $G_n(x)$ (and $\tilde{G}_n(x)$) is symmetric if $n \equiv 1 \bmod 4$ (i.e. if $G_n(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0$, $a_{d-k} = a_k$ for all k).

We also know that

$$L_n(x) := \prod_{j\in S_n}(x - \zeta_{2n'}^{\ j}) = C_n(x^2) - s'x\sqrt{n}D_n(x^2),$$

$$L_n(-x) = \Phi_n(sx^2)/L_n(x) = C_n(x^2) + s'x\sqrt{n}D_n(x^2),$$

where $S_n =$

$$\begin{cases} \{j \mid 0 < j < 2n',\ (j, n') = 1,\ \left(\frac{j}{n}\right) = (-1)^j\}, & \text{if } n \equiv 1 \bmod 4, \\ \{j \mid 0 < j < 2n',\ (j, n') = 1,\ \left(\frac{n}{j}\right) = 1\}, & \text{otherwise.} \end{cases}$$

From the definition, $L_n(x)$ is also symmetric for any n and $C_n(x), D_n(x) \in \boldsymbol{Z}[x]$.

Furthermore, for $|x| \leq 1$, we get the following. (see [1])

$$(3)\quad G_n(x) = \sqrt{\Phi_n(x)}\exp\left(-\frac{s\sqrt{sn}}{2}f_n(x)\right),$$

$$(4)\quad L_n(x) = \sqrt{\Phi_n(sx^2)}\exp(-s'\sqrt{n}g_n(x)),$$

where $f_n(x) = \sum_{j=0}^{\infty}\left(\frac{j}{n}\right)\frac{x^j}{j}$ and

$$g_n(x) = \sum_{j=0}^{\infty}\left(\frac{n}{2j+1}\right)\frac{x^{2j+1}}{2j+1}.$$

As we are interested in the real quadratic fields, suppose $n \equiv 1 \bmod 4$. We want the value of $L_n(1)$. Since $2g_n(x\sqrt{s})/\sqrt{s} = f_n(x) - f_n(-x)$, we need the values of $f_n(1)$ and $f_n(-1)$ in order to get the value of $g_n(1)$.

Step 1. Note that $f_n(1) = L(1, \chi)$ where $\chi(j) = \left(\frac{j}{n}\right)$ is the real, non-trivial Dirichlet character. So the value $f_n(1)$ is related to the class number $h(D)$ of the quadratic field $\boldsymbol{Q}(\sqrt{n})$ with discriminant $D = n'$.