A Higher-dimensional Analogue of Carlitz-Drinfeld Theory

By Hideki TANUMA

Department of Mathematics, Tokyo Institute of Technology

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1995)

The purpose of this paper is to generalize the arguments of Carlitz and Drinfeld to the higher-dimensional case by giving some analogies of special functions like the Carlitz exponential, the zeta function, the gamma functions, and the modular forms. This is a résumé of my master thesis at University of Tokyo, March 1994, and the details will be published elsewhere.

In the paper of Kapranov [6], the method of the completion is given and the higher-dimensional version of the zeta function is defined. So we apply the idea of Kapranov to define some analogues of the special functions other than the zeta function and deduce the properties of these functions.

1. An analogue of Carlitz exponential. Let $A = A_n = \boldsymbol{F}_q[T_1, \dots, T_n]$ be the polynomial ring over finite field in n variables and $k = k_n = \boldsymbol{F}_q(T_1, \dots, T_n)$ be its field of quotients. The ring A is embedded discretely into the complete topological field $K = K_n = \boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_n)\!)$ with t_n-adic valuation when we set

$$T_1 = \frac{t_{n-1}}{t_n},\ T_2 = \frac{t_{n-2}}{t_n}, \dots, T_{n-1} = \frac{t_1}{t_n},\ T_n = \frac{1}{t_n}.$$

Let $C = C_n = \hat{\bar{K}}$ be the completion of the algebraic closure of the field K and for any $\boldsymbol{F}_q$-lattice Λ over C we define the function $e_\Lambda : C \to C$

$$e_\Lambda(z) = z \prod_{\lambda \in \Lambda - O} \left(1 - \frac{z}{\lambda}\right)$$

where we call any discrete submodule 'lattice'.

The function e_Λ satisfies the following properties.

(1) e_Λ is entire.

(2) e_Λ is $\boldsymbol{F}_q$-linear and Λ-periodic.

(3) e_Λ has simple zeroes at the points of Λ, and no further zeroes.

(4) if Λ, $\Lambda' = c\Lambda (c \in C^*)$ are similar lattices, then $ce_\Lambda(z) = e_{\Lambda'}(cz)$.

(5) The derivative satisfies $e'_\Lambda(z) = 1$.

We define the power series $\phi_a^\Lambda(z)$ by $e_\Lambda(az) = \phi_a^\Lambda(e_\Lambda(z))$. In the higher-dimentional case, for an A-module Λ of finite rank, we have $^\#(a^{-1}\Lambda/\Lambda) = \infty$ for any $a \in A - \boldsymbol{F}_q$, and $\phi_a^\Lambda(z)$ is not a polynomial like in the one-dimentional case.

In the two-dimensional case, we have the following theorem.

Theorem 1. *Let $A = A_2 = \boldsymbol{F}_q[X, Y]$ and (X, Y^i) be the ideal of A generated by X and Y^i. Then the coefficients of the series*

$$e_A(Xz) = \sum_{i=0}^{\infty} l_i e_A(z)^{q^i}$$

are written as

$$l_O = X,\ l_i = X^{q^i} \sum_{0 \le j_1 < j_2 \dots < j_i} \tau_{j_1} \tau_{j_2}^q \cdots \tau_{j_i}^{q^{i-1}} \ (i > 0),$$
$$\tau_i = -e_{(X, Y^{i+1})}(Y^i)^{1-q}$$

and their valuations are

$$v(l_i) = q^i + (q-1) \sum_{j=0}^{i-1} \sum_{k=1}^{j} q^{j + \frac{k(k-1)}{2}}.$$

2. The analogue of zeta function. The Goss zeta function was generalized to the case of $A = A_n = \boldsymbol{F}_q[T_1, \dots, T_n]$ by Kapranov [6]. We recall the construction.

We start with the definition of the term 'monic'. For $a \in K$, let $a^{(1)}$ be the element of $\boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_{n-1})\!)$ such that

$$a = a^{(1)} t_n^{v(a)} + a',\ v(a') > v(a).$$

Similarly, $a^{(2)} \in \boldsymbol{F}_q(\!(t_1)\!) \dots (\!(t_{n-2})\!)$ can be derived from $a^{(1)}$ and finally we get an element $a^{(n)} \in \boldsymbol{F}_q$. In this case, we call an element a 'monic' iff $a^{(n)} = 1$. The set of monic elements is closed under multiplication and this definition is compatible with the old one for $A = A_1$.

For any natural integer s the series

$$\zeta_A(s) = \sum_{\text{monic } a \in A} \frac{1}{a^s}$$

is convergent because the point set $\{a^{-s} \mid a \in A - 0\}$ has at most finite points in neighborhood of 0 and non-Archimedean property shows this. In addition to this, $A = A_n$ is also an UFD as in the case of one-dimensional, then the above sum has the Euler product

$$\prod_{\text{monic irred. } \wp \in A} (1 - \wp^{-s})^{-1}.$$