On the Kloosterman-sum Zeta-function

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The aim of the present paper is to show that given the spectral resolution of the hyperbolic Laplacian there is an argument which leads us fairly quickly to Kuznetsov's spectral expansion [1, (7.26)] of the Kloosterman-sum zeta-function $Z_{m,n}(s)$. Since his formula is equivalent to his much quoted trace formula [1, Theorem 2], our argument provides the latter with a more accessible proof. To prove his result on $Z_{m,n}(s)$ Kuznetsov developed a quite ingenious argument of transforming the inner-product of two Poincaré series into a series of J-Bessel functions integrated with respect to their orders, and applied various averaging technique to extract the defining series of $Z_{m,n}(s)$. Thus, though powerful and impressive, his argument inevitably depended heavily on the theory of Bessel functions as is well indicated by his use of exotic identities such as that of Gegenbauer [3, p. 138 (1)]. We shall dispense with those heavy machineries altogether.

Before starting our discussion it should be worth remarking that though we restrict ourselves to the case of the full modular group $\Gamma = SL(2, \mathbf{Z})$ it is apparent that we do not lose any generality.

Now, the Kloosterman-sum zeta-function is defined as

$$Z_{m,n}(s) = (2\pi\sqrt{mn})^{2s-1} \sum_{l=1}^{\infty} S(m, n; l) l^{-2s},$$

$$\left(m, n > 0, \operatorname{Re}(s) = \sigma > \frac{3}{4}\right),$$

where $S(m, n; l)$ is the Kloosterman sum

$$\sum_{\substack{h=1 \\ (h,l)=1}}^{l} \exp(2\pi i(mh + n\bar{h})/l), \ h\bar{h} \equiv 1 \bmod l.$$

We are going to extract this series from Poincaré series. To this end we take the same initial step as Kuznetsov's or rather that of Selberg [2]. Thus, we introduce the Poincaré series

$$P_m(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\operatorname{Im}\gamma(z))^s \exp(2\pi i m\gamma(z)), \ \sigma > 1, \tag{1}$$

where m is a positive integer, $z = x + iy$ $(y > 0)$ and Γ_∞ the stabilizer of the point at infinity. We have the well-known Fourier expansion

$$\begin{aligned} P_m(z, s) = {} & y^s \exp(2\pi i m z) + \\ & y^{1-s} \sum_{n=-\infty}^{\infty} \exp(2\pi i n x) \sum_{l=1}^{\infty} l^{-2s} S(m, n; l) \\ & \times \int_{-\infty}^{\infty} \exp\left(-2\pi i n y\xi - \frac{2\pi m}{l^2 y(1 - i\xi)}\right) \\ & \times (1 + \xi^2)^{-s} d\xi, \end{aligned} \tag{2}$$

which is equivalent to regrouping the summands in (1) according to the double coset decomposition $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$. Thus Weil's estimate for $S(m, n; l)$ yields that $P_m(z, s)$ is regular in the region $\sigma > \frac{3}{4}$, where we have also the bound $P_m(z, s) \ll y^{1-\sigma}$, providing y is not too small. This means in particular that $P_m(z, s)$ is in the Hilbert space $L^2(\mathscr{F}, d\mu)$ when $\sigma > \frac{3}{4}$; here $\mathscr{F}$ is the fundamental region of Γ and $d\mu$ the Poincaré metric as usual. We should note that Weil's bound for $S(m, n; l)$ is not mandatory but a bound like Estermann's classical estimate is sufficient for our purpose. At any event the above implies that we may apply the spectral decomposition to the inner product.

$$\langle P_m(\cdot, s_1), P_n(\cdot, \overline{s_2})\rangle = \int_{\mathscr{F}} P_m(z, s_1) \overline{P_n(z, \overline{s_2})} d\mu(z).$$

To state the decomposition we let $\left\{\lambda_j = \kappa_j^2 + \frac{1}{4}; \kappa_j > 0, j \geq 1\right\} \cup \{0\}$ stand for the discrete spectrum of the hyperbolic Laplacian acting on $L^2(\mathscr{F}, d\mu)$. Also let ψ_j be an eigen-form corresponding to λ_j so that it has the Fourier expansion

$$\psi_j(z) = \sqrt{y} \sum_{n \neq 0} \rho_j(n) K_{i\kappa_j}(2\pi |n| y) \exp(2\pi i n x),$$

where K_ν is the K-Bessel function of order ν. We may assume that the set $\{\psi_j\}$ forms an orthonormal system. Then we have, for $\sigma_1, \sigma_2 > \frac{3}{4}$