On Homology and Cohomology of Lie Superalgebras with Coefficients in Their Finite-Dimensional Representations

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In this paper we discuss explicit calculations of homology and cohomology of a Lie superalgebra. Complete results fore $\mathfrak{gl}(1,1)$ and $\mathfrak{sl}(2,1)$ are given in case the dimensions of representations are finite. Our result implies that for any $n \in \boldsymbol{Z}_{\geq 0}$, there exists a finite-dimensional irreducible $\mathfrak{g}$-module V such that $\mathbf{H}^n(\mathfrak{g}, V) \neq \{0\}$, contrary to the case of finite-dimensional Lie algebras. This means that the Poincaré duality, which is proved by S.Chemla [1] under a certain restrictive condition, does not hold in general in our case. For definitions and notations we mainly follow Kac [6].

1. Generalities. Homology groups $\mathbf{H}_n(\mathfrak{g}, V)$ of a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with coefficients in its representation space V are defined similarly as for a Lie algebra (cf. [7, p. 283]) and they can be obtained as $\operatorname{Ker} \partial_{n-1} / \operatorname{Im} \partial_n$ in the following complex (B, ∂):

$$0 \leftarrow B_0 \overset{\partial_0}{\leftarrow} B_1 \overset{\partial_1}{\leftarrow} B_2 \overset{\partial_2}{\leftarrow} B_3 \overset{\partial_3}{\leftarrow} \cdots, \quad B_n = \wedge^n \mathfrak{g} \otimes V,$$

$$\begin{aligned} &\partial_{n-1}(X_1 \wedge \cdots \wedge X_n \otimes v) \\ &= \sum_{i=1}^{n} (-1)^{i+\eta'_i} X_1 \wedge \cdots \hat{i} \cdots \wedge X_n \otimes X_i v \\ &\quad + \sum_{k<l} (-1)^{k+l+\eta_k+\eta_l+\xi_k\xi_l} [X_k, X_l] \\ &\qquad \wedge X_1 \wedge \cdots \hat{k} \cdots \hat{l} \cdots \wedge X_n \otimes v, \end{aligned}$$

where $X_i \in \mathfrak{g}$ homogeneous, $v \in V$, $\xi_i = |X_i| := \deg X_i$, $\eta_i = \xi_i(\xi_1 + \cdots + \xi_{i-1})$, $\eta'_i = \xi_i(\xi_{i+1} + \cdots + \xi_n)$, and the symbol $\hat{i}$ indicates a term X_i to be omitted (cf. [8]). The Grassmann algebra $\wedge \mathfrak{g}$ here is defined as the quotient of the tensor algebra of $\mathfrak{g}$ by a two-sided ideal generated by $\{X \otimes Y + (-1)^{|X||Y|} Y \otimes X \mid X, Y \in \mathfrak{g};$ homogeneous$\}$ and it is a $\mathfrak{g}$-module through a natural action:

$$\begin{aligned} &X \cdot (X_1 \wedge \cdots \wedge X_n) \\ &= \sum (-1)^{|X|(\xi_1 + \cdots + \xi_{i-1})} X_1 \wedge \cdots \wedge [X, X_i] \wedge \cdots \wedge X_n. \end{aligned}$$

Then B_n's are $\mathfrak{g}$-modules with $\rho_n(X)(\theta \otimes v) = X\theta \otimes v + (-1)^{|X||\theta|} \theta \otimes Xv$ ($X \in \mathfrak{g}$, $\theta = X_1 \wedge \cdots \wedge X_n \in \wedge^n \mathfrak{g}$, $|\theta| = \xi_1 + \cdots + \xi_n$, $v \in V$). This action commutes with the derivation ∂, that is, $X \circ \partial_n = \partial_{n-1} \circ X$.

We appeal to the following lemmas to calculate the homology and the cohomology.

Lemma 1. *Let $\mathfrak{q}$ be a subalgebra of $\mathfrak{g}$ such that its natural representation $\rho_n|_{\mathfrak{q}}$ on the n-th chain B_n are all semisimple. Then, the homology $\mathbf{H}_n(\mathfrak{g}, V)$ can be obtained from a subcomplex $(B^{\mathfrak{q}}, \partial|_{B^{\mathfrak{q}}})$, where the n-th chain $B_n^{\mathfrak{q}}$ for $B^{\mathfrak{q}}$ is the subspace of $\mathfrak{q}$-invariants in B_n.*

The space $V^* := \operatorname{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$ has a natural $\mathfrak{g}$-module structure.

Lemma 2 (Duality). *Let $\mathfrak{g}$ be a Lie superalgebra and V a $\mathfrak{g}$-module. Assume that $\mathfrak{g}$ and V are both finite-dimensional, then there are $\mathfrak{g}$-module isomorphisms between homology groups and cohomology groups as*

$$\mathbf{H}^n(\mathfrak{g}, V^*) \cong \mathbf{H}_n(\mathfrak{g}, V)^*.$$

2. Case of $\mathfrak{gl}(1,1)$. Fix a basis of the Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(1, 1)$ as follows:

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The elements H and C generate a Cartan subalgebra, which is equal to the even part $\mathfrak{g}_{\bar{0}}$ of $\mathfrak{g}$ in this simplest case. Put $\mathfrak{g}_1 = \boldsymbol{C}X$ and $\mathfrak{g}_{-1} = \boldsymbol{C}Y$. Then the odd part is $\mathfrak{g}_{\bar{1}} = \mathfrak{g}_1 + \mathfrak{g}_{-1}$, and this gives a $\boldsymbol{Z}$-grading of $\mathfrak{g}$ together with $\mathfrak{g}_0 = \mathfrak{g}_{\bar{0}}$. Let $L(\Lambda) := \boldsymbol{C}v_0$ be a one-dimensional representation of $\mathfrak{g}_{\bar{0}}$ given by $Hv_0 = \lambda v_0$, $Cv_0 = cv_0$ ($\lambda, c \in \boldsymbol{C}$) and Λ denote a pair (λ, c). For a subalgebra $\mathfrak{p} := \mathfrak{g}_{\bar{0}} + \mathfrak{g}_1$, we extend $L(\Lambda)$ as a $\mathfrak{p}$-module through a trivial action of X. Then the induced module $\bar{V}(\Lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathfrak{p}} L(\Lambda)$ defines a representation of $\mathfrak{g}$. $\bar{V}(\Lambda)$ is irreducible if and only if $c \neq 0$.

We calculate the homology $\mathbf{H}_n(\mathfrak{g}, \bar{V}(\Lambda))$, which is isomorphic to $\mathbf{H}_n(\mathfrak{p}, L(\Lambda))$ by Shapiro's lemma on induced modules (cf. [7]). Put $X^{(k)} = X \wedge X \wedge \cdots \wedge X \in \wedge^k \mathfrak{g}$ and