Fermat Varieties of Hodge-Witt Type

By Keisuke TOKI

Department of Mathematics, Chuo University

(Communicated by Shokichi IYANAGA, M. J. A., March 13, 1995)

§1. Introduction. Let X be a smooth projective variety over a perfect field k of characteristic $p > 0$, and W be the ring of Witt vectors on k. If X is of Hodge-Witt in all degrees in the sense of (4.6) in Chapitre IV of Illusie-Raynaud [4], that is, if Hodge-Witt cohomology groups $H^j(X, W\Omega_X^i)$ are of finite type over W for all (i, j), then we say that X is *of Hodge-Witt type.* E.g. if X is a curve, X is of Hodge-Witt type (cf. Serre [7], Chapter II of Illusie [2]). When X is a smooth complete intersection in a projective space, we know, through Suwa [9], that X is of Hodge-Witt type if the "niveau de Hodge" of X in the sense of Deligne [1] or Rapoport [6] is at most one. If $V_n(m)$ means a hypersurface of degree m in an $(n+1)$-dimensional projective space $\boldsymbol{P}^{n+1}$, each $V_n(m)$ for "$n > 0$, $m = 2$", "$n = 2$, $m = 3$", "$n = 3$, $m = 3$", "$n = 3$, $m = 4$", "$n = 5$, $m = 3$" is of the "niveau de Hodge" ≤ 1 according to Rapoport [6], 2, Table 1. Moreover we know that if X is *ordinary* in the sense of (4.12) in Chapitre IV of Illusie-Raynaud [4] then X is of Hodge-Witt type.

We are concerned with the smooth hypersurface S of degree $m > 0$ defined by an equation:

$$a_0x_0^m + a_1x_1^m + \cdots + a_{n+1}x_{n+1}^m = 0$$

(the a_i are in k, and not 0) over a finite field k of characteristic $p > 0 (p \nmid m)$ in $\boldsymbol{P}^{n+1}$ of which homogeneous coordinates are $x_0, x_1, \ldots, x_{n+1}$. Then, over an algebraic closure of k, the hypersurface S is isomorphic to the Fermat variety $F_{n,m,p}$ of dimension $n > 0$, degree $m > 0$ defined by

$$x_0^m + x_1^m + \cdots + x_{n+1}^m = 0$$

in $\boldsymbol{P}^{n+1}$

To show that S is of Hodge-Witt type, it is sufficient to show that $F_{n,m,p}$ is of this type in degree n by Suwa [9].

Now we consider the Fermat variety $F_{n,m,p}$ with $\{n, m, p\} (n > 0, m > 0, p \nmid m)$. From what we have said above, we know the followings:

Case (1) $\{n, m, p\} = \{n, 1, p\}$ or $\{n, 2, p\}$;

$F_{n,1,p}$ and $F_{n,2,p}$ are ordinary and hence of Hodge-Witt type.

Case (2) $\{n, m, p\} = \{1, m, p\}$;

$F_{1,m,p}$ is of Hodge-Witt type.

Case (3) $\{n, m, p\}$ *with* $p \equiv 1 \bmod m$;

$F_{n,m,p}$ is ordinary and hence of Hodge-Witt type.

Case (4)

$$\{n, m, p\} = \begin{cases} \{2,3, p\} & \text{with } p \equiv 2 \bmod 3, \\ \{3,3, p\} & \text{with } p \equiv 2 \bmod 3, \\ \{3,4, p\} & \text{with } p \equiv 3 \bmod 4, \\ \{5,3, p\} & \text{with } p \equiv 2 \bmod 3; \end{cases}$$

$F_{2,3,p}$, $F_{3,3,p}$, $F_{3,4,p}$ and $F_{5,3,p}$ are of Hodge-Witt type.

In addition we have obtained the following result through Suwa's criterion (see §2).

Theorem. *Let the triplet $\{n, m, p\}$ of integers $n > 1$, $m > 2$, and a prime number p with $p \nmid m$ and $p \not\equiv 1 \pmod m$ be given. Then we have the following assertion:*

$F_{n,m,p}$ is of Hodge-Witt type

if and only if

$\{n, m, p\}$ is in the above case (4) or in the case (5) "$n = 2$, $m = 7$ and $p \equiv 2,4 \pmod 7$".

The assertion for $n = 2$, $m \geq 4$ in the Theorem has been conjectured by N. Suwa.

The author wishes to express his hearty thanks to Prof. N. Suwa, who has communicated to him this conjecture and the known results recalled at the beginning of this paper.

§2. The set $\mathcal{W}$. Let p be a prime number. And let the triplet $\{n, m, p\}$ of integers with $n > 0$, $m > 0$, $p \nmid m$ be given. For $w = (w_0, w_1, \ldots, w_{n+1}) \in \boldsymbol{Z}^{n+2}$, let the integer $|w|$ be defined by

$$|w| = \sum_{j=0}^{n+1} w_j.$$

Moreover, we set

$$\mathcal{W} = \{w \in \boldsymbol{Z}^{n+2}; 0 < w_j < m (j = 0,1,2, \ldots, n+1), \\ |w| \equiv 0 \bmod m\},$$

$$\mathcal{W}_i = \{w \in \mathcal{W}; |w| = (i+1)m\} (i = 0,1,2, \ldots).$$

Then we have