Explicit Representation of Fundamental Units of Some Quadratic Fields

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 13, 1995)

1. Introduction. Explicit form of the fundamental unit of real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ is not well-known except for real quadratic fields of Richaud-Degert type.

In this paper, for all real quadratic fields $\boldsymbol{Q}(\sqrt{d})$ such that d is a positive square-free integer congruent to 1 mod 4 and the period k_d in the continued fraction expansion of the quadratic irrational number $\omega_d = (1+\sqrt{d})/2$ in $\boldsymbol{Q}(\sqrt{d})$ is equal to 3, we describe explicitly T_d, U_d in the fundamental unit $\varepsilon_d = (T_d + U_d\sqrt{d})/2(>1)$ of $\boldsymbol{Q}(\sqrt{d})$ and d itself by using two parameters l, r appearing in the continued fraction expansion of ω_d. Finally, as an application of this theorem, we provide a result on class number one problem for real quadratic fields and on Yokoi's invariant n_d.

For the set $I(d)$ of all quadratic irrational numbers in $\boldsymbol{Q}(\sqrt{d})$, we say that α in $I(d)$ is reduced if $\alpha > 1$, $-1 < \alpha' < 0$ (α' is the conjugate of α with respect to $\boldsymbol{Q}$), and denote by $R(d)$ the set of all reduced quadratic irrational numbers in $I(d)$. Then, it is well-known that any number α in $R(d)$ is purely periodic in the continued fraction expansion and the denominator of its modular automorphism is equal to the fundamental unit ε_d of $\boldsymbol{Q}(\sqrt{d})$, and that the norm of ε_d is $(-1)^{k_d}$ (see, for example, [2] p. 205, 215). Moreover the continued fraction with period k is generally denoted by $[a_0, \overline{a_1, \ldots, a_k}]$, and $[x]$ means the greatest integer not greater than x.

Now, for any square-free positive integer d congruent to 1 mod 4, we put $d = a^2 + b$, $0 < b \leq 2a$ $(a, b \in \boldsymbol{Z})$. Here, since $\sqrt{d} - 1 < a < \sqrt{d}$, both integers a and b are uniquely determined by d. Then, our main theorem is as follows:

Theorem. *For a square-free positive integer d congruent to 1 mod 4, we assume $k_d = 3$. Then, in the case that a is odd,*

$$\omega_d = [(a+1)/2, \overline{l, l, a}],$$

and

$$(T_d, U_d) = ((l^2+1)^2 r + l(l^2+3),\ l^2+1)$$

hold for two positive integers l, r such that $a = (l^2+1)r + l$.

Moreover in this case, it holds

$$d = (l^2+1)^2 r^2 + 2l(l^2+3)r + l^2 + 4.$$

In the case that a is even,

$$\omega_d = [a/2, \overline{1, 1, a-1}],\ (T_d, U_d) = (2a, 2) \text{ and } d = a^2 + 1$$

hold.

In order to prove this theorem, we need several lemmas.

Lemma 1. *For a square-free positive integer $d > 5$ congruent to 1 modulo 4, we put $\omega = (1+\sqrt{d})/2$, $q_0 = [\omega]$ and $\omega_R = q_0 - 1 + \omega$. Then $\omega \notin R(d)$, but $\omega_R \in R(d)$ holds. Moreover for the period k of ω_R, we get $\omega_R = [\overline{2q_0 - 1, q_1, \ldots, q_{k-1}}]$ and $\omega = [q_0, \overline{q_1, \ldots, q_{k-1}, 2q_0 - 1}]$. Furthermore, let $\omega_R = (P_k\omega_R + P_{k-1})/(Q_k\omega_R + Q_{k-1}) = [2q_0 - 1, q_1, \ldots, q_{k-1}, \omega_R]$ be a modular automorphism of ω_R, then the fundamental unit ε_d of $\boldsymbol{Q}(\sqrt{d})$ is given by the following formula:*

$$\varepsilon_d = (T + U\sqrt{d})/2 > 1,$$

$$T = (2q_0 - 1)Q_k + 2Q_{k-1},\ U = Q_k,$$

where Q_i is determined by $Q_0 = 0$, $Q_1 = 1$, $Q_{i+1} = q_iQ_i + Q_{i-1}$, $(i \geq 1)$.

Proof. Denote by Nm and Tr the norm and the trace respectively. Then $\omega_R = (2q_0 - 1 + \sqrt{d})/2$ belongs to $I(d)$, because ω_R is a root of the equation $X^2 - T_r(\omega_R)X + Nm(\omega_R) = 0$ and the discriminant of this equation is $Tr(\omega_R)^2 - 4Nm(\omega_R) = d$. Moreover since $\omega_R' = [\omega] - \omega > -1$ and $2q_0 - 1 < \sqrt{d}$, we get $0 > \omega_R' > -1$. Hence ω_R belongs to $R(d)$. Since $[\omega_R] = [[\omega] - 1 + \omega] = 2q_0 - 1$ and ω_R is purely periodic, ω_R and ω have expansions described in this Lemma respectively. Since $Q_k\omega_R + Q_{k-1}$ is the fundamental unit of $\boldsymbol{Q}(\sqrt{d})$ with norm $(-1)^k$ (see, for example, [2] p. 215), $\varepsilon_d = Q_k\{q_0 - 1 + (1+\sqrt{d})/2\} + Q_{k-1} = \{(2q_0 - 1)Q_k + 2Q_{k-1} + Q_k\sqrt{d}\}/2$. Thus, the proof of Lemma 1 was completed.

We apply the recurrence formula in [1] to ω_R, and get useful parameters essentially connected with partial quotients of the continued