On Some Exceptional Rational Maps

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§1. Introduction. Throughout this paper let $\hat{C} = C \cup \{\infty\}$ be the Riemann sphere and $R : \hat{C} \to \hat{C}$ be a rational map of degree $d \geq 2$. Periodic orbits are one of the most important objects to study in the theory of dynamical systems. By definition a point z_0 is an *n-periodic point* if

$$R^n(z_0) := \overbrace{R \circ \cdots \circ R}^{n}(z_0) = z_0$$
$$\text{and } R^k(z_0) \neq z_0 \text{ for } k < n$$

and we call the set $\{z_0, R(z_0), R^2(z_0), \ldots, R^{n-1}(z_0)\}$ an *n-periodic orbit.* It is a fundamental problem to ask if there exists at least one n-periodic orbit of R for each natural number n, or equivalently, in which case rational maps fail to have n-periodic orbits for some n. For this problem the following result is well known.

Theorem ([1]). *Let R be a rational map of degree $d \geq 2$. If R has no n-periodic orbit, then the pair (d, n) is either* (2,2), (2,3), (3,2) *or* (4,2).

From this theorem one can say that "most" rational maps have at least one n-periodic orbit for any n and a rational map which does not satisfy this property is rather "exceptional". In [1] Baker also constructed concrete examples for each pair of (d, n) above to show that the exceptional maps really exist. Now, how many kinds of such maps exist? In this paper we answer this question, that is, we investigate whole these exceptional rational maps and completely classify them up to the conjugation by Möbius transformations. Also we give an inequality that holds among the numbers of n-periodic orbits and give a lower estimate of the number of p-periodic orbits in the case where p is prime.

For the definitions of the concepts we do not mention here, see for example [2].

§2. Normal forms and a key lemma. In order to classify all the exceptional rational maps, it is sufficient to determine all the conjugacy classes and give each representative element. For this purpose we construct normal forms for each $d(= 2,3, \text{ or } 4)$ and find the representatives in it. If R is an exceptional rational map, all the 2-periodic orbits (or 3-periodic orbits in the case of $(d, n) = (2,3)$) are considered to be degenerate to some fixed points (or equivalently 1-periodic orbits). Therefore it is convenient to construct normal forms which specify some fixed points. Since one can easily see that each exceptional rational map has at least two distinct fixed points, we may assume that 0 and ∞ are such points by the conjugacy of the Möbius transformation which maps these two distinct fixed points to 0 and ∞. With one more suitable Möbius transformation for each cases, we obtain the following normal forms. Here, for example, $m(0; R)$ denotes the multiplier of R at the fixed point $z = 0$.

Proposition. *Every rational map of degree 2, 3,* or *4 which admits at least two distinct fixed points is conjugate to the following normal forms, respectively.*

(i) $d = 2 \quad R(z) = \dfrac{z^2 + bz}{az + 1}, \quad a = m(\infty; R),$
$b = m(0; R).$

(ii) $d = 3 \quad R(z) = \dfrac{z^3 + cz^2 + ez}{az^2 + bz + 1},$
$a = m(\infty; R), \quad e = m(0; R).$

(iii) $d = 4 \quad R(z) = \dfrac{z^4 + ez^3 + fz^2 + gz}{az^3 + bz^2 + cz + 1},$
$a = m(\infty; R), \quad g = m(0; R).$

The next lemma is a key for solving our problem.

Lemma. *Suppose that $R(z_0) = z_0$. Let $mul(z_0; R)$ denotes the multiplicity of the fixed point $z = z_0$. For a fixed natural number n,*

(i) *if $\{m(z_0; R)\}^n \neq 1$, then $mul(z_0; R) = mul(z_0; R^n)$.*

In the case of $\{m(z_0; R)\}^n = 1$,

(ii) *if $m(z_0; R) = 1$, then $mul(z_0; R) = mul(z_0; R^n) > 1$.*

(iii) *If $m(z_0; R)$ is a primitive t-th root of unity, then $t \mid n$ and $mul(z_0; R) = 1$ and $mul(z_0;$*