The Rankin's *L*-function and Heegner Points for General Discriminants

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In this paper we show some results on the relation between the first derivatives of the Rankin's L-series of certain modular forms at $s = 1$ and the heights for certain divisors on the Jacobian of the modular curve $X_0(N)$. These divisors consist of Heegner points whose orders have conductor f. The proof of our main result consists of a long complicated "analytic" computation (see [5]). This generalizes the "analytic" part of the influential work of Gross and Zagier [4], which has established a relation between the first derivatives of the Rankin's L-series of certain modular forms at $s = 1$ and the height pairings for squarefree discriminants prime to N. Their results can be applied to give the proof of a special case of the Birch-Swinnerton-Dyer conjecture, and are needed to complete Goldfeld's solution of Gauss conjecture for the class number of imaginary quadratic fields. Kolyvagin [6] has used the result of [4] in his proof of the finiteness of the Tate-Shafarevich groups of certain modular elliptic curves over $\boldsymbol{Q}$. J. van der Lingen [7] has calculated "algebraically" the local Néron-Tate height pairings "at non-archimedean places" for certain divisor on $X_0(N)$ consisting of Heegner points whose orders have general discriminants prime to N. He has found explicit formulas for these local height pairings at non-archimedian places. But it is difficult to compare his formulas and ours.

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§1. Let us begin with recalling some definitions. Let K be an imaginary quadratic field with the fundamental discriminant D_0, and $\mathcal{O}$ an order with the discriminant $D = D_0 f^2$ of the conductor f in K. Let $h_f = \#\,\mathrm{Pic}(\mathcal{O})$ and $u = \#\,(\mathcal{O}^\times/\{\pm 1\})$. We have $u = 1$ unless $D = -3, -4$, in which cases $u = 3, 2$, respectively.

We say $x = (E \to E')$ is a "Heegner point" of discriminant D on $X_0(N)$ if both of the elliptic curves E and E' have complex multiplication by $\mathcal{O}$. Such a point exists if and only if D is congruent to a square modulo $4N$; equivalently every prime divisor of N splits or is ramified in K. If one Heegner point exists on $X_0(N)$, then there are $2^s \cdot h_f$ Heegner points with $s = \#\{p \mid N\}$ which are all rational over the "ring class field" $K_f = K(j(E))$ of K. Those Heegner points are attached to a fixed integral ideal $\mathfrak{n}(\mathrm{N}(\mathfrak{n}) = N)$ of $\mathcal{O}$ with $\mathcal{O}/\mathfrak{n} \simeq \boldsymbol{Z}/N\boldsymbol{Z}$. They are permuted simply-transitively by the abelian group $W \times \mathrm{Gal}(K_f/K)$ and those actions on Heegner points can be described explicitly, where $W \cong (\boldsymbol{Z}/2\boldsymbol{Z})^s$ is the group of Atkin-Lehner involutions and $\mathrm{Gal}(K_f/K)$ the Galois group of K_f/K, which is canonically isomorphic to the class group $\mathrm{Pic}(\mathcal{O})$ of $\mathcal{O}$ via the Artin reciprocity map (see [1]).

In this paper, D is not assumed to be square free nor relatively prime to N on $X_0(N)$, but assume throughout that the conductor f is relatively prime to N. Fix a Heegner point x of discriminant D; then the class of the divisor $c = (x) - (\infty)$ defines an element in $J(K_f)$, where (∞) denotes the sum of cusps at infinity on $X_0(N)$, which is defined over $\boldsymbol{Q}$, where J is the Jacobian of $X_0(N)$.

Let $f(z) = \sum_{n \geq 1} a(n) e^{2\pi i n z}$ be an element in the vector space of newforms of weight 2 on $\Gamma_0(N)$, $\varepsilon(\cdot) = \left(\frac{D}{\cdot}\right)$ the Kronecker Symbol and $r_{\mathcal{A}}(n)$ the number of integral invertible ideals of $\mathcal{O}$ of norm n in the class $\mathcal{A}$. We define the Rankin's L-function associated to the newform $f(z)$ and the ideal class $\mathcal{A}$ by

$$L_{\mathcal{A}}(f, s) = L^{(N)}(2s - 2k + 1, \varepsilon) \cdot \sum_{n \geq 1} a(n) r_{\mathcal{A}}(n) n^{-s},$$

where

$$L^{(N)}(2s - 2k + 1, \varepsilon) = \sum_{\substack{n \geq 1 \\ (n, DN) = 1}} \varepsilon(n) n^{-2s+2k-1}.$$

The series $L^{(N)}$ is the Dirichlet L-function of ε at