# On Certain Dirichlet Series Obtained by the Product of Eisenstein Series and a Cusp Form

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**§1.** Let $\mathscr{H}$ be an upper half plane ,$\Gamma = SL_2(\boldsymbol{Z})$ and $\Gamma_\infty$ be the stabilizer of the cusp $i\infty$ of $\Gamma$. The real analytic Eisenstein series $E(z, \alpha)$ is defined by

$$E(z, \alpha) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\gamma z)^\alpha \text{ for } \mathrm{Re}\, \alpha > 1.$$

We put $E^*(z, \alpha) = \xi(2\alpha)E(z, \alpha)$ where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ and $\zeta(s)$ is the Riemann zeta function. It is well known that the function $E^*(z, \alpha)$ has a holomorphic continuation to all $\alpha$ except for simple poles at $\alpha = 0$ and 1 and satisfies the functional equation $E^*(z, \alpha) = E^*(z, 1 - \alpha)$. The Fourier expansion is given by

$$E^*(z, \alpha) = \xi(2\alpha)y^\alpha + \xi(2 - 2\alpha)y^{1-\alpha} + 2\sum_{n \neq 0} |n|^{1/2-\alpha}\sigma_{2\alpha-1}(|n|)y^{1/2}K_{\alpha-1/2}(2\pi|n|y)e^{2\pi i n x}.$$

Here, $K_\nu(z)$ denotes the so-called modified Bessel function and $\sigma_\nu(n) = \sum_{d|n} d^\nu$.

In [5], Vinogradov and Takhtadzhyan studied the classical additive divisor problem through the spectral theory of automorphic functions. Namely they showed that the main term of the integral

$$\int_0^\infty \int_0^1 |E^*(z, 1/2)|^2 y^s e^{2\pi i k z} \frac{dxdy}{y^2}$$

is $\pi^{-s}\Gamma(s/2)^4\Gamma(s)^{-1}\sum_{n=1}^\infty d(n)d(n+k)n^{-s}$ and got the growth order of the last Dirichlet series by the spectral theory of automorphic functions.

**§2.** We consider here the product of the Eisenstein series and a cusp form and derive the corresponding Dirichlet series. Let $f(z)$ be a Maass wave form with the parity $\varepsilon_f$ and its Fourier expansion be given by

$$f(z) = \sum_{n \neq 0} \rho(n)y^{1/2}K_{i\kappa}(2\pi|n|y)e^{2\pi i n x}.$$

We assume that $\rho(n) = O(|n|^{\eta_0})$ for some $\eta_0 > 0$. Up to now, it is known that $\eta_0 \leq 5/28$. (cf. [1])

For a natural integer $k$, we define

$$I_k(s; \alpha, f) = \int_0^\infty \int_0^1 E^*(z, \alpha)f(z)y^s e^{2\pi i k x}\frac{dxdy}{y^2}.$$

**Lemma 1.** *Let $s$ be a complex number. If* $\mathrm{Re}\, s$ *is sufficiently large, we have*

$$\begin{aligned} I_k(s; \alpha, f) &= (4\pi^s\Gamma(s))^{-1}\Gamma\Big(\frac{s+\alpha-1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s+\alpha-1/2-i\kappa}{2}\Big)\Gamma\Big(\frac{s-\alpha+1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\alpha+1/2-i\kappa}{2}\Big) \\ &\times \Big\{\sum_{m=1}^\infty \frac{\sigma_{2\alpha-1}(m+k)\rho(m)}{m^{s+\alpha-1/2}} F\Big(\frac{s+\alpha-1/2+i\kappa}{2}, \\ &\qquad \frac{s+\alpha-1/2-i\kappa}{2}; s; 1-\Big(\frac{m+k}{m}\Big)^2\Big) \\ &+ \varepsilon_f \sum_{m=1, m\neq k}^\infty \frac{\sigma_{2\alpha-1}(|m-k|)\rho(m)}{m^{s+\alpha-1/2}} \\ &\times F\Big(\frac{s+\alpha-1/2+i\kappa}{2}, \frac{s+\alpha-1/2-i\kappa}{2}; s; \\ &\qquad 1-\Big(\frac{m-k}{m}\Big)^2\Big)\Big\} \\ &+ \varepsilon_f\rho(k)\varphi_0(s; \alpha), \end{aligned}$$

*where $F(\alpha, \beta, \gamma; x)$ is the hypergeometric function and*

$$\begin{aligned} \varphi_0(s, \alpha) &= \frac{\xi(2\alpha)}{4(\pi k)^{s+\alpha-1/2}} \\ &\times \Gamma\Big(\frac{s+\alpha-1/2+i\kappa}{2}\Big)\Gamma\Big(\frac{s+\alpha-1/2-i\kappa}{2}\Big) \\ &+ \frac{\xi(2-2\alpha)}{4(\pi k)^{s-\alpha+1/2}}\Gamma\Big(\frac{s-\alpha+1/2+i\kappa}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\alpha+1/2-i\kappa}{2}\Big). \end{aligned}$$

This lemma can be shown by the Fourier expansions of $E^*(z, \alpha)$, $f(z)$ and the following integral formula:

$$\begin{aligned} \int_0^\infty K_\nu(ny)K_\mu(my)\,dy &= 2^{s-3}m^{-s-\nu}n^\nu\Gamma(s)^{-1} \\ &\times \Gamma\Big(\frac{s+\mu+\nu}{2}\Big)\Gamma\Big(\frac{s+\mu-\nu}{2}\Big)\Gamma\Big(\frac{s-\mu+\nu}{2}\Big) \\ &\times \Gamma\Big(\frac{s-\mu-\nu}{2}\Big) \\ &\times F\Big(\frac{s+\nu+\mu}{2}, \frac{s+\nu-\mu}{2}; s; 1-(n/m)^2\Big). \end{aligned}$$

(cf. [2] p. 93 (36))