Note on Siegel-Eisenstein Series

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1. Siegel–Eisenstein series. In this paper, we will treat two types of Eisenstein series and give some remarks. Let $\boldsymbol{H}_n$ be the Hermitian upper half space of degree n, namely, the domain consisting of all complex square matrices of size n such that the Hermitian imaginary part $\Im(Z) := (2i)^{-1}(Z - \bar{Z}^T)$ is positive definite. Here $\bar{Z}^T$ is the transpose, complex conjugate matrix of Z. The Siegel upper half space $\boldsymbol{S}_n := \{Z \in \boldsymbol{H}_n \mid Z^T = Z\}$ is a submanifold of $\boldsymbol{H}_n$. If $Z \in \boldsymbol{S}_n$, then $I(Z) := \Im(Z)$ is exactly equal to the imaginary part of Z. Consider an imaginary quadratic field K of discriminant d_K. The ring of integers in K is denoted by $\mathcal{O} = \mathcal{O}_K$. The Hermitian modular group of degree n associated with K is defined as:

$$\Gamma_n(K) := \left\{M \in SL_{2n}(\mathcal{O}) \mid \bar{M}^T J_n M = J_n, J_n = \begin{pmatrix} 0_n & E_n \\ -E_n & 0_n \end{pmatrix}\right\}.$$

The Siegel modular group of degree n is defined as $\Gamma_n := Sp_n(\boldsymbol{Z})$. Let $[\Gamma_n, k]$ (resp. $[\Gamma_n(K), k]$) be the vector space of holomorphic Siegel modular forms (resp. Hermitian modular forms) of weight k for Γ_n (resp. $\Gamma_n(K)$).

Let us consider the Eisenstein series of the following two types:

(*SP Case*)

$$E_k^{(n)}(Z, s) := \det I(Z)^s \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_{n,0} \backslash \Gamma_n} \det(CZ + D)^{-k} \mid \det(CZ + D) \mid^{-2s}, \ Z \in \boldsymbol{S}_n$$

(*SU Case*)

$$E_{k,K}^{(n)}(Z, s) := \det \Im(Z)^s \sum_{\left(\begin{smallmatrix} * & * \\ C & D \end{smallmatrix}\right) \in \Gamma_n(K)_0 \backslash \Gamma_n(K)} \det(CZ + D)^{-k} \mid \det(CZ + D) \mid^{-2s}, \ Z \in \boldsymbol{H}_n$$

Here k is an even integer and $\Gamma_{n,0}$ (resp. $\Gamma_n(K)_0$) is the subgroup of Γ_n (resp. $\Gamma_n(K)$) consisting of the elements $M = \begin{pmatrix} A & B \\ 0_n & D \end{pmatrix}$ in Γ_n (resp. $\Gamma_n(K)$). It is known that $E_k^{(n)}(Z, s)$ (resp. $E_{k,K}^{(n)}(Z, s)$) is convergent for $\mathrm{Re}(s) > (n + 1 - k)/2$ (resp. $\mathrm{Re}(s) > (2n - k)/2$). Moreover, they can be continued as meromorphic functions in s to the whole complex plane. The analytic properties of these Eisenstein series were successfully studied by Shimura [5] and Weissauer [6]. In fact, Shimura found the following results.

Theorem 1 (Shimura). (1) (*SP Case*) *$E_{\frac{n-1}{2}}^{(n)}(Z, s)$ has at most a simple pole at $s = 1$. The residue at $s = 1$ is π^{-n} times an element f in $\left[\Gamma_n, \dfrac{n-1}{2}\right]$ with rational Fourier coefficients.*

(2) (*SU Case*) *$E_{n-1,K}^{(n)}(Z, s)$ has at most a simple pole at $s = 1$. The residue at $s = 1$ is π^{-n} times an element f in $[\Gamma_n(K), n - 1]$ with rational Fourier coefficients.*

Remark 1. The definition of Eisenstein series in [5] is slightly different from our definition. The Eisenstein series Shimura treated were $\det I(Z)^{-\frac{s}{2}} E_k^{(n)}\left(Z, \frac{s}{2}\right)$ (SP Case) and $\det \Im(Z)^{-\frac{s}{2}} E_{k,K}^{(n)}\left(Z, \frac{s}{2}\right)$ (SU Case) in our notation.

2. A residue formula. Our purpose is to specify the modular forms f in Theorem 1. The first result is as follows:

Theorem 2. (1) *For any even, positive integer k such that $k < \dfrac{n+1}{2}$, $E_k^{(n)}(Z, s)$ is holomorphic in s at $s = 0$ and $E_k^{(n)}(Z, 0)$ defines an element of $[\Gamma_n, k]$ with rational Fourier coefficients.*

(2) *Assume that the class number of K is 1. For any even, positive integer k such that $k < n$, $E_{k,K}^{(n)}(Z, s)$ is holomorphic in s at $s = 0$ and $E_{k,K}^{(n)}(Z, 0)$ defines an element of $[\Gamma_n(K), k]$ with rational Fourier coefficients.*

A proof of (1) was already given in Weissauer [6]. Another proof is found by using results of Arakawa [1] and Mizumoto [3].

Here we must introduce the following notation:

$$\xi(s) := \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

$$\xi(s ; \chi_K) := \pi^{-\frac{s}{2}} \mid d_K \mid^{\frac{s}{2}} \Gamma\left(\frac{s+1}{2}\right) L(s ; \chi_K),$$