A Characterization of Regularly Almost Periodic Minimal Flows

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Abstract: In this paper we shall prove two theorems: Firstly, a minimal flow is regularly almost periodic if and only if it is almost automorphic and the dimension of the set of eigenvalues is 1. Secondly, a minimal flow is pointwise regularly almost periodic if and only if it is equicontinuous and the dimension of the set of eigenvalues is 1.

§1. Introduction. Let X be a metric space with metric d_X. Z, Q, R and C denote the set of integers, rational numbers, real numbers and complex numbers, respectively. A continuous mapping $\pi : X \times R \to X$ is said to be a *flow on* (*a phase space*) X if π satisfies the following conditions:

(1) $\pi(x, 0) = x$ for $x \varepsilon X$.

(2) $\pi(\pi(x, t), s) = \pi(x, t + s)$

for $x \varepsilon X$ and $t, s \varepsilon R$.

For $A \subset X$ and $B \subset R$, we denote the set $\{\pi(x, t) ; x \varepsilon A, t \varepsilon B\}$ by $\pi(A, B)$. The closure of $A \subset X$ is denoted by $\bar{A}$. For $x \varepsilon X$ we denote the orbit through $x \varepsilon X$ by $O_\pi(X)$, that is, $O_\pi(x) = \pi(x, R)$. $M \subset X$ is called *an invariant set of* π if $O_\pi(x) \subset M$ for each $x \varepsilon M$. The restriction of π to an invariant set M of π is denoted by $\pi \mid M$. A non-empty compact invariant set $M \subset X$ is said to be *a minimal set of* π if we have $\overline{O_\pi(x)} = M$ for each $x \varepsilon M$. If X is itself a minimal set of π, we say that π is *a minimal flow on* X. π is said to be *equicontinuous* if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(\pi(x, t), \pi(y, t)) < \varepsilon$ for $d_X(x, y) < \delta$ and $t \varepsilon R$.

Let π be a minimal flow on a compact metric space X. $x \varepsilon X$ is called *a regularly almost periodic point* if for each $\varepsilon > 0$ there exists an $\alpha > 0$ such that $\pi(x, n\alpha) \varepsilon U_\varepsilon(x)$ for $n \varepsilon Z$, where $U_\varepsilon(x) = \{z \varepsilon X ; d_X(x, z) < \varepsilon\}$. The set of regularly almost periodic points is denoted by $R(\pi)$. If $R(\pi) \neq \phi$, we say that π is *regularly almost periodic*. If $R(\pi) = X$, we say that π is *pointwise regularly almost periodic*. $x \varepsilon X$ is said to be *an almost automorphic point* if $\pi(x, \tau_n) \to y$ as $n \to \infty$ for some sequence $\{\tau_n\} \subset R$ implies that $\pi(y, -\tau_n) \to x$ as $n \to \infty$. The set of almost automorphic points is denoted by $A(\pi)$. If $A(\pi) \neq \phi$, we say that π is *almost automorphic*. We can easily see that $R(\pi)$ and $A(\pi)$ are invariant sets of π. $\lambda \varepsilon R$ is said to be *an eigenvalue of* π if there exists a continuous mapping $\chi_\lambda : X \to K = \{\xi \varepsilon C ; |\xi| = 1\}$ such that $\chi_\lambda(\pi(x, t)) = \chi_\lambda(x)\exp(i\lambda t)$ for $x \varepsilon X$ and $t \varepsilon R$. In this case, χ_λ is called *an eigenfunction belonging to* λ. The set of eigenvalues of π is denoted by $\Lambda(\pi)$. We can easily verify that $\Lambda(\pi)$ is a countable subgroup of the additive group R.

$\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \varepsilon R$ are said to be *rationally independent* if $r_1\alpha_1 + r_2\alpha_2 + \ldots\ldots + r_n\alpha_n = 0 (r_i \varepsilon Q)$ implies $r_1 = r_2 = \ldots\ldots = r_n = 0$. We say that a countable subset A of R has *dimension* n if there exist $\alpha_1, \alpha_2, \ldots\ldots\ldots, \alpha_n \varepsilon R$, which are rationally independent, such that we have $a = r_1\alpha_1 + r_2\alpha_2 + \ldots\ldots + r_n\alpha_n (r_i \varepsilon Q)$ for each $a \varepsilon A$. The dimension of $A \subset R$ is denoted by $\dim A$.

In [4] regularly almost periodic minimal flows are discussed for discrete phase group. In this paper we characterize them for one parameter flows. In section 2 we shall show the following theorems.

Theorem 1. *Let* π *be a minimal flow on a compact metric space* X. *Then* π *is regularly almost periodic if and only if it is almost automorphic and* $\dim \Lambda(\pi) = 1$.

Theorem 2. *Let* π *be a minimal flow on a compact metric space* X. *Then* π *is pointwise regularly almost periodic if and only if it is equicontinuous and* $\dim \Lambda(\pi) = 1$.

§2. Proofs of Theorems. In this section we shall prove Theorems 1 and 2. In order to prove them, we need several propositions.

Let π and ρ be flows on compact metric spaces X and Y, respectively. A continuous map-