Copula Fields and their Applications

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0. Introduction. The construction of stochastic processes from a family of consistent probability measures can be done by Kolmogorov's extention theorem (see [1]).

But the construction of stochastic processes from a family of nonconsistent probability measures can not always be done.

In this paper we propose the following problems and give the answers.

(P1). For any $T > 0$ and any family of Borel probability measures $\{\rho(t, dx)\}_{0 \leq t \leq T}$ on R^d, construct a R^d-valued Markov process $\{X(t)\}_{0 \leq t \leq T}$ on a probability space (Ω, B, P) such that

(0.1) $P(X(t) \in dx) = \rho(t, dx)$ for all $t \in [0, T]$.

(P2). For any $T > 0$, any family of Borel probability measures $\{\rho(t, dx)\}_{0 \leq t \leq T}$ on R^d, and any Borel probability measure $\mu(dxdy)$ on R^{2d} for which $\int_{y \in R^d} \mu(dxdy) = \rho(0, dx)$ and for which $\int_{x \in R^d} \mu(dxdy) = \rho(T, dy)$, construct a R^d-valued reciprocal process (see [5]) $\{X(t)\}_{0 \leq t \leq T}$ on a probability space $(\Omega, \boldsymbol{B}, P)$ such that

(0.2) $P(X(t) \in dx) = \rho(t, dx)$ for all $t \in [0, T]$,

(0.3) $P(X(0) \in dx, X(T) \in dy) = \mu(dxdy)$.

Main idea is that of copula in the multivariate analysis (see [2,7,8]). We give the definition of a **copula field**, extending the idea, directly, to the path space.

We also give the applications to the stochastic control. **(P1)** is related to the stochastic quantizations (see [6] and references therein).

1. Copula fields and one dimensional case. In this section we show how to construct a real valued stochastic process from a family of Borel probability measures on $\boldsymbol{R}$, extending directly the idea of copula, to the path space. We also give the definition of the **copula field**. In this section we denote by $\boldsymbol{I}$ the parameter space.

Let us give the definition of a copula for a real valued stochastic process which is well defined from [7], Theorems 6.2.4, 6.2.5.

Definition 1.1. For any real valued stochastic process $\{X(t)\}_{t \in I}$ on a probability. space $(\Omega, \boldsymbol{B}, P)$, the family $(C_A^X(u_1, \cdots, u_{\#(A)}))_{A \subset I, \#(A) < \infty}$ of copulas which satisfies the following is called a **copula for** $\{X(t)\}_{t \in I}$; for any $A = \{t_1^A, \cdots, t_{\#(A)}^A\} \subset I$ and any $x_1, \cdots, x_{\#(A)} \in R$

(1.1) $P(X(t_1^A) \leq x_1, \cdots, X(t_{\#(A)}^A) \leq x_{\#(A)}) =$
$$C_A^X(F_{t_1^A}^X(x_1), \cdots, F_{t_{\#(A)}^A}^X(x_{\#(A)})),$$
where we put $F_t^X(x) = P(X(t) \leq x)$.

Before we give the definition of a copulas field for a real valued stochastic process, let us give some notations. Denote by $DF(R)$ the set of all continuous distribution functions on R. For $F \in DF(R)$, we can define the functions $F^*(u)$ $(0 \leq u \leq 1)$ by the following; put

$$F^*(0) \equiv \begin{cases} \max\{x; F(x) = 0\} & \text{if } 0 \in Range(F), \\ -\infty & \text{if } 0 \notin Range(F), \end{cases}$$

(1.2) $F^*(u) \equiv \min\{x ; F(x) = u\}$ for $0 < u < 1$,

$$F^*(1) \equiv \begin{cases} \min\{x; F(x) = 1\} & \text{if } 1 \in Range(F), \\ \infty & \text{if } 1 \notin Range(F) \end{cases}$$

(see [7], p. 49). Put $DF(R)^* \equiv \{F^*; F \in DF(R)\}$; $DF(R)_I \equiv \{\{F_t\}_{t \in I}; F_t \in DF(R)(t \in I)\}$; $DF(R)_I^* \equiv \{\{F_t^*\}_{t \in I}; F_t \in DF(R)(t \in I)\}$.

Definition 1.2. For any real valued stochastic process $\{X(t; \omega)\}_{t \in I, \omega \in \Omega}$ on a probability space $(\Omega, \boldsymbol{B}, P)$, the **copula field** $\{\boldsymbol{C}^X(\boldsymbol{F}^*; \omega)(t)\}_{t \in I, \boldsymbol{F}^* \in DF(R)_I^*, \omega \in \Omega}$ **for** $\{X(t; \omega)\}_{t \in I, \omega \in \Omega}$ is defined as follows; for all $t \in I$, $\boldsymbol{F}^* = \{F_s^*\}_{s \in I} \in DF(R)_I^*$, and $P - a.a.\omega$

(1.3) $\boldsymbol{C}^X(\boldsymbol{F}^*; \omega)(t) = F_t^*(F_t^X(X(t; \omega)))$.

When there is no confusion, we simply denote the copula field by $\boldsymbol{C}^X(\boldsymbol{F}^*)(t)$, omitting ω.

Remark 1.1. The copula for a real valued stochastic process $\{X(t)\}_{t \in I}$ is uniquely determined if and only if $F_t^X(x)$ is continuous in $x \in R$ for all $t \in I$. Copula field for a real valued stochastic process is unique. F^* is a quasi-inverse of F (see [7], p. 49), and our choice in (1.2) is convenient as we show in the next proposition whose proof is omitted.