On the Generators of the Mapping Class Group of a 3-dimensional Handlebody

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Let V be a handlebody of genus $g(\geq 2)$ and let $T = \partial V$. Let $\mathcal{M}_V$, $\mathcal{M}_T$ be the mapping class groups of V, T, respectively. (For the definition of a mapping class group, see [3].) It is well known that $\mathcal{M}_T$ is isomorphic to the outer automorphism group of $\pi_1(T)$.

We have the injection $\nu : \mathcal{M}_V \to \mathcal{M}_T$ by letting the restriction $f \mid T : T \to T$ correspond to each homeomorphism $f : V \to V$.

In this paper we seek the generators of $\nu(\mathcal{M}_V)(\subset \mathcal{M}_T)$ which are as simple as possible as the products of the generators defined by Lickorish in [4].

Let α_i, β_i $(i = 1, \cdots, g)$, $\nu_i (i = 1, \cdots, g-1)$ be the isotopy classes of the Dehn twists about the simple loops shown in the Fig. 1.

We shall prove the following theorem.

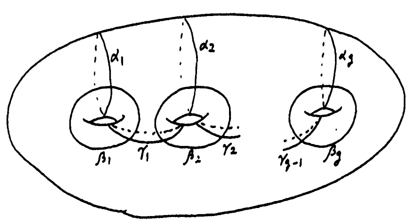

Fig. 1

Theorem. $\nu(\mathcal{M}_V)$ is generated by α_1, $\beta_1\alpha_1^2\beta_1$, $\beta_i\alpha_i\gamma_i\beta_i$, $\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}$ $(i = 1, \cdots, g-1)$.

Proof. Let a_i, b_i $(i = 1, \cdots, g)$ be the generators of the fundamental group of the surface T as shown in the Fig. 2.

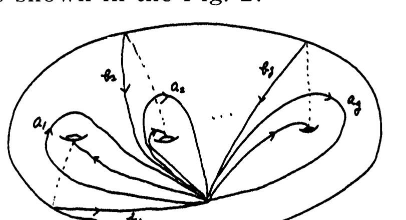

Fig. 2

$(\pi_1(T) \simeq \langle a_i, b_i (i = 1, \cdots, g) \mid s_1^{-1}s_2^{-1} \cdots s_g^{-1} = 1\rangle$, where $s_i = a_i^{-1}b_i^{-1}a_ib_i$, $i = 1, \cdots, g$.)

By Suzuki [1], $\mathcal{M}_V$ is generated by the isotopy classes of τ_1, ω_1, θ_{12}, ξ_{12}, ρ_{12} and ρ. The induced automorphisms of $\pi_1(T)$ are given by:

$$\nu(\tau_1)\ \begin{cases} a_1 \to b_1^{-1}a_1 \\ a_i \to a_i\ (i = 2, \cdots, g) \\ b_i \to b_i (i = 1, \cdots, g), \end{cases}$$

$$\nu(\omega_1)\ \begin{cases} a_1 \to a_1^{-1}s_1^{-1} \\ a_i \to a_i (i = 2, \cdots, g) \\ b_1 \to a_1^{-1}b_1^{-1}a_1 \\ b_i \to b_i (i = 2, \cdots, g) \end{cases}$$

$$\nu(\theta_{12})\ \begin{cases} a_1 \to a_1s_2^{-1}a_1^{-1} \\ a_i \to a_i (i = 2, \cdots, g) \\ b_2 \to a_2b_2a_1^{-1}b_1a_1b_2^{-1}a_2^{-1}b_2 \\ b_i \to b_i (i \neq 2) \end{cases}$$

$$\nu(\xi_{12})\ \begin{cases} a_1 \to b_1a_1b_2^{-1}s_2a_1^{-1}b_1^{-1}a_1 \\ a_2 \to a_2b_2a_1^{-1}b_1^{-1}a_1b_2^{-1} \\ a_i \to a_i (i = 3, \cdots, g) \\ b_i \to b_i (i = 1, \cdots, g), \end{cases}$$

$$\nu(\rho_{12})\ \begin{cases} a_1 \to s_1^{-1}a_2s_1 \\ a_2 \to a_1 \\ a_i \to a_i (i = 3, \cdots, g) \\ b_1 \to s_1^{-1}b_2s_1 \\ b_2 \to b_1 \\ b_i \to b_i (i = 3, \cdots, g) \end{cases}$$

$$\nu(\rho)\ \begin{cases} a_i \to a_{i+1}\ (i = 1, \cdots, g-1) \\ a_g \to a_1 \\ b_i \to b_{i+1} (i = 1, \cdots, g-1) \\ b_g \to b_1 \end{cases}$$

First we observe that each element stated in the theorem is actually an element of $\nu(\mathcal{M}_V)$. By [2], an element of $\mathcal{M}_T$ is in $\nu(\mathcal{M}_V)$ if and only if, by the induced automorphism of $\pi_1(T)$, $\langle b_1, \cdots, b_g\rangle$ is mapped in the normal subgroup generated by $\langle b_1, \cdots, b_g\rangle$.

Now the induced automorphisms of α_1, $\beta_1\alpha_1^2\beta_1$, $\beta_i\alpha_i\gamma_i\beta_i$, $\beta_{i+1}\alpha_{i+1}\gamma_i\beta_{i+1}$ are given by