Cubic Hyper-equisingular Families of Complex Projective Varieties. II

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This is a continuation of our previous paper [4], which will be referred to as Part I in this note. We inherit the notation and terminology of it.

§3. Variations of mixed Hodge structure.

3.1 Theorem. *Let* $\mathscr{X}.\xrightarrow{a.}\mathscr{X}\xrightarrow{\pi}M$ *be an* n*-cubic* $(n\geq 1)$ *hyper-equisingular family of complex projective varieties, parametrized by a complex manifold* M. *We define* $R^{\ell}_{\boldsymbol{Z}}(\pi):=R^{\ell}\pi_*\boldsymbol{Z}_{\mathscr{X}}$ *(modulo torsion)* $(0\leq\ell\leq 2(\dim\mathscr{X}-\dim M))$, $R^{\ell}_{\boldsymbol{Q}}(\pi):=R^{\ell}_{\boldsymbol{Z}}(\pi)\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ *and* $R^{\ell}_{\mathscr{O}}(\pi):=R^{\ell}\pi_*(\pi^{\cdot}\mathscr{O}_M)\simeq R^{\ell}\pi_*(DR^{\cdot}_{\mathscr{X}/M})$, *where* $\pi^{\cdot}\mathscr{O}_M$ *is the topological inverse of the structure sheaf of* M *by the map* $\pi:\mathscr{X}\to M$ *and* $DR^{\cdot}_{\mathscr{X}/M}$ *the cohomological relative de Rham complex of the family* $\pi:\mathscr{X}\to M$. *Then there exist a family of increasing sub-local systems* $\boldsymbol{W}$ *(weight filteration) on* $R^{\ell}_{\boldsymbol{Q}}(\pi)$ *and a family of decreasing holomorphic subbundles* $\boldsymbol{F}$ *(Hodge filteration) on* $R^{\ell}_{\mathscr{O}}(\pi)$ *such that*

(i) there are spectral sequences

$$_{W}E_1^{p,q}\simeq\bigoplus_{|\alpha|=p+1}R^q\pi_{\alpha*}\boldsymbol{Q}_{\mathscr{X}_\alpha}\Rightarrow$$
$$_{W}E_\infty^{p,q}=Gr^{W}_{-p}(R^{p+q}_{\boldsymbol{Q}}(\pi)),$$
$$_{F}E_1^{p,q}\simeq\boldsymbol{R}^q\pi_*(s(a_{1.*}\Omega^{p}_{\mathscr{X}./M})[1])\Rightarrow$$
$$_{F}E_\infty^{p,q}=Gr^{p}_{F}(R^{p+q}_{\mathscr{O}}(\pi))$$

with $_{W}E_2^{p,q}={}_{W}E_\infty^{p,q}$, $_{F}E_1^{p,q}={}_{F}E_\infty^{p,q}$,

(ii) $(R^{\ell}_{\boldsymbol{Z}}(\pi),\boldsymbol{W}[\ell],\boldsymbol{F})$ *defines mixed Hodge strucutre at each point* $t\in M$, *where* $\boldsymbol{W}[\ell]$ *denotes the shift of the filteration degree to the right* by ℓ, i.e., $\boldsymbol{W}[\ell]_q:=\boldsymbol{W}_{q-\ell}$, and

(iii) (the Griffiths transversality)

$$\nabla\mathscr{F}^p\subset\Omega^1_M\otimes\mathscr{F}^{p-1},$$

where ∇ denotes the Gauss-Mannin connection on $R^{\ell}_{\mathscr{O}}(\pi)$.

Outline of the proof. (i), (ii): By Theorem 2.1 and Theorem 2.2 in [4], we have an isomorphism

$$\pi^{\cdot}\mathscr{O}_M\approx DR^{\cdot}{}_{\mathscr{X}/M}\approx s(a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M})[1]$$

in $D^+(\mathscr{X},\boldsymbol{C})$, where $a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M}$ is the n-cubic object of complexes of $\boldsymbol{C}$-vector spaces coming from $\Omega^{\cdot}_{\mathscr{X}./M}$, and $s(a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M})$ is its associated single complex (cf. Part I, [1, Exposé I,6]). By this isomorphism we have

$$R^{\ell}_{\mathscr{O}}(\pi):=R^{\ell}\pi_*(\pi^{\cdot}\mathscr{O}_M)\simeq\boldsymbol{R}^{\ell}\pi_*(s(a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M})[1]).$$

To compute the hyper-direct image $\boldsymbol{R}^{\ell}\pi_*(s(a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M})[1])$, we shall use the fine resolution $\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M}$ of $\Omega^{\cdot}_{\mathscr{X}./M}$, where $\mathscr{A}^{r,s}_{\mathscr{X}_\alpha/M}$ are the sheaves of C^∞ relative differential forms of type (r,s) on $\mathscr{X}_\alpha(\alpha\in\square_n)$. Then the natural homomorphism

$$s(a_{1.*}\Omega^{\cdot}_{\mathscr{X}./M})[1]\to s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1]$$

is an isomorphism in $D^+(\mathscr{X},\boldsymbol{C})$, where $\mathrm{tot}\,\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}$ is the single complex associated to the double complex $\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}$ for each $\alpha\in\square_n$. Since $s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1]$ is π_*-acyclic, we have

$$R^{\ell}_{\mathscr{O}}(\pi)\simeq H^{\ell}(\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1]).$$

We define an increasing filteration $\boldsymbol{W}=\{W_q\}$ and a decreasing one $F=\{F^q\}$ on the single complex $L:=\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1]$ by

$$W_{-q}(\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1])$$
$$:=\sigma_{|\alpha|\geq q+1}\pi_*s(a_{1\alpha*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M})\quad(q\geq 0)\text{ and}$$
$$F^{p}(\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1])$$
$$:=\sigma_{k\geq p}\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{k\cdot}_{\mathscr{X}./M})[1]\quad(p\geq 0),$$

where $\sigma_{|\alpha|\geq q+1}\pi_*s(a_{1\alpha*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}_\alpha/M}):=\sigma_{\geq q}(L)$ if we put $L:=\pi_*s(a_{1.*}\mathrm{tot}\mathscr{A}^{\cdot\cdot}_{\mathscr{X}./M})[1]$. ($\sigma_{\geq q}$: *stupid filteration*). Notice that the filteration W is defined over $\boldsymbol{Q}$. We calculate the spectral sequence associated to these filterations, abutting to $R^{\ell}_{\mathscr{O}}(\pi)$. Since (L_t,W,F) is a *cohomological mixed Hodge complex* in the sense of Deligne for any $t\in M$ (for definition see [1, (8.1.6)]), the spectral sequence $\{E_r(L_t,W),d_r\}$ degenerates at the E_2-terms and the one associated to F degenerates at the E_1-terms ([2, p.48, Théorème 3.2.1 (Deligne), (vi), (v)]). The assertions (i) and (ii) follow from this.

(iii): We take a point $o\in M$ and put $X_\alpha:=(\pi\cdot a_\alpha)^{-1}(o)$, $X:=\pi^{-1}(o)$. By the definition of an n-cubic hyper-equisingular family $\mathscr{X}.\xrightarrow{a.}\mathscr{X}\xrightarrow{\pi}M$, it is analytically locally trivial. Hence, schrinking M sufficiently small around o, we are allowed to assume that there is a system of Stein coverings $\mathscr{U}_\alpha:=\{U_i^{(\alpha)}\}_{i\in\Lambda_\alpha}$ of X_α $(\alpha\in\square_n^+)$, which is subject to the following requirements:

(1) for each pair (α,β) of elements of $\mathrm{Ob}(\square_n^+)$ with $\alpha\to\beta$ in $\square_n^+$, there is a map $\lambda_{\alpha\beta}:\Lambda_\beta\to\Lambda_\alpha$ such that