Cubic Hyper-equisingular Families of Complex Projective Varieties. I

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Introduction. The purpose of this note is to outline a recent result of the author's study on *cubic hyper-equisingular families of complex projective varieties,* from which there naturally arise variations of mixed Hodge structure. In order to define such families we use *cubic hyper-resolutions* of complex projective varieties due to V. Navarro Aznar, F. Guillén *et al.*, [1]. The initial motivation for this study was to describe the variation of mixed Hodge structure which might be expected to arise from a *locally trivial* family of projective varieties with *ordinary singularities* (cf. [3], [4]). Details will be published elsewhere.

§1. Cubic hyper-equisingular families of complex projective varieties. We denote by $\boldsymbol{Z}$ the integer ring.

1.1 Definition. For $n \in \boldsymbol{Z}$ with $n \geq 0$ the *augmented n-cubic category,* denoted by $\square_n^+$, is defined to be a category whose objects $\mathrm{Ob}(\square_n^+)$ and the set of homomorphisms $\mathrm{Hom}_{\square_n^+}(\alpha, \beta)$ ($\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_n) \in \mathrm{Ob}(\square_n^+)$) are given as follows:

$$\mathrm{Ob}(\square_n^+) := \{\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_n) \in \boldsymbol{Z}^{n+1} \mid 0 \leq \alpha_i \leq 1 \text{ for } 0 \leq i \leq n\},$$

$$\mathrm{Hom}_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \to \beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i \leq \beta_i \text{ for } 0 \leq i \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$ we understand $\square_{-1}^+$ to be the punctual category $\{*\}$, i. e., the category consisting of one point. For $n \geq 0$ the *n-cubic category,* denoted by $\square_n$, is defined to be the full subcategory of $\square_n^+$ with $\mathrm{Ob}(\square_n) = \mathrm{Ob}(\square_n^+) - \{(0, \ldots, 0)\}$. Notice that $\mathrm{Ob}(\square_n^+) - \{(0, \ldots, 0)\}$ (resp. $\mathrm{Ob}(\square_n)$) can be considered as a finite ordered set whose order is defined by $\alpha \leq \beta \Leftrightarrow \alpha \to \beta$ for $\alpha, \beta \in \mathrm{Ob}(\square_n^+)$ (resp. $\mathrm{Ob}(\square_n)$).

1.2 Definition. A $\square_n^+$-object (resp. $\square_n$-object) of a category $\mathscr{C}$ is a contravariant functor $X_\cdot^+$ (resp. $X_\cdot$) from $\square_n^+$ (resp. $\square_n$) to $\mathscr{C}$. It is also called an *augmented n-cubic object of $\mathscr{C}$* (resp. an *n-cubic object of $\mathscr{C}$*).

1.3 Definition. Let $X_\cdot$, $Y_\cdot$ be $\square_n^+$-objects of a category $\mathscr{C}$. We define a morphism $\Phi_\cdot : X_\cdot \to Y_\cdot$ to be a natural transformation from the functor $X_\cdot$ to the one $Y_\cdot$ over the identity functor $\mathrm{id}: \square_n^+ \to \square_n^+$.

Let $X_\cdot$ be an n-cubic object of $\mathscr{C}$ ($n \geq 0$), X a (-1)-object of $\mathscr{C}$. We denote by $X \times \square_n$ the n-cubic object defined by $(X \times \square_n)(\alpha) = X$ for every $\alpha \in \square_n$. An *augmentation of $X_\cdot$ to X* is a morphism from $X_\cdot$ to $X \times \square_n$. We may think of an n-cubic object of $\mathscr{C}$ with an augmentation to X as an augmented n-cubic object of $\mathscr{C}$. Conversely, an augmented n-cubic object $X_\cdot^+ : (\square_n^+)^\circ \to \mathscr{C}$ of $\mathscr{C}$ can be identified with an n-cubic object $X_\cdot := X^+_{\cdot|\square_n} : (\square_n)^\circ \to \mathscr{C}$ of $\mathscr{C}$ with an augmentation to $X^+_{(0,\ldots,0)}$. In the following we shall interchangeably use an augmented n-cubic object of $\mathscr{C}$ and an n-cubic object of $\mathscr{C}$ with an augmentation.

1.4 Definition. For a $\square_n^+$-complex projective variety $X_\cdot$, a contravariant functor $Y_\cdot$ from $\square_1^+$ to the category of $\square_n^+$-complex projective varieties is called a *2-resolution of* $X_\cdot$ if $Y_\cdot$ is defined by a cartesian square of morphisms of $\square_n^+$-complex projective varieties

$$\begin{array}{ccc} Y_{11\cdot} & \longrightarrow & Y_{01\cdot} \\ \downarrow & & \downarrow f \\ Y_{10\cdot} & \longrightarrow & Y_{00\cdot}, \end{array} \tag{1.1}$$

which satisfies the following conditions:

(i) $Y_{00\cdot} = X_\cdot$,

(ii) $Y_{01\cdot}$ is a smooth $\square_n^+$-complex projective variety, i.e., a contravariant functor from $\square_n^+$ to the category of smooth complex projective varieties,

(iii) the horizontal arrows are closed immersion of $\square_n^+$-complex projective varieties,

(iv) f is a proper morphism between $\square_n^+$-complex projective varieties, and

(v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in \mathrm{Ob}(\square_n^+)$.

We think of the cartesian square in (1.1) as a morphism from the $\square_{n+1}^+$-complex projective variety $Y_{1\cdot\cdot}$ to the one $Y_{0\cdot\cdot}$ and write it as $Y_{1\cdot\cdot} \to Y_{0\cdot\cdot}$. For a 2-resolution $Z_\cdot$ of $Y_{1\cdot\cdot}$, we define the