Radon Transform on Distributions

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Abstract: In the literature there are three apparently different definitions of the Radon transform on various spaces of distributions: Gelfand-Graev's, Helgason's and Ludwig's. In this paper a new definition of the Radon transform on the space of the tempered distributions is given and it is proved that, properly understood, the earlier definitions are all equivalent to the new one. A constructive description (a characterization) of the space of test functions is given. A simple method for studying the range of the Radon transform on some spaces of distributions is described.

Key words: Tomography; Radon transform; distributions.

1. Introduction. In the literature there are three apparently different definitions of the Radon transform R on the spaces of distributions. The first, given by I. Gelfand and M. Graev, (GG definition), (see [1]), Helgason's definition (H definition) based on the duality formula ([2]), and Ludwig's definition (L definition) ([3]).

In this paper these definitions are analyzed, their common features are demonstrated; a new definition is given and it is proved that, properly understood, the earlier definitions are all equivalent to the new one and, therefore, they are all equivalent; a constructive description of the space of test functions is given; a simple method for studying the range of R on distributional spaces is described.

We use the notations and some results from the forthcoming monograph [4]. In [5]-[6] some properties of the Radon transform on various spaces of distributions are given and H definition is used. In [7] and [8] some related results are obtained.

2. The three definitions of R. 2.1. Let us introduce the standard notations: $\mathcal{S}' := \mathcal{S}'(\boldsymbol{R}^n)$ is the space of tempered distributions on $\mathcal{S} := \mathcal{S}(\boldsymbol{R}^n)$, $\mathcal{D}'$ is the space of distributions on $\mathcal{D} := C_0^\infty(\boldsymbol{R}^n)$, and $\mathcal{E}'$ is the space of distributions on $\mathcal{E} := C^\infty(\boldsymbol{R}^n)$. The Radon transform R on $\mathcal{S}$ is defined by the formula:

$$Rf := \tilde{f} := \int_{l_{\alpha p}} f(x)\,ds, \tag{1}$$

where $l_{\alpha p} = \{x : \alpha \cdot x = p\}$ is a plane, ds is the Lebesgue measure on this plane, $p \in \boldsymbol{R}$, $\alpha \in S^{n-1}$, the unit sphere in $\boldsymbol{R}^n$.

Let $\tilde{f} := \mathcal{F} f := \int_{\boldsymbol{R}^n} \exp(ix \cdot \xi) f(x)\,dx$, $Fh := \int_{-\infty}^{\infty} \exp(ip\lambda) h(\alpha, p)\,dp$, $\xi = \lambda\alpha$. An elementary corollary of the Fourier slice theorem is [4]:

$$R = F^{-1}\mathcal{F} \text{ on } \mathcal{S}. \tag{2}$$

Let $\mathcal{S}_e := \mathcal{S}_e(Z)$ be the Schwartz space of even functions $h(-\alpha, -p) = h(\alpha, p)$ on $Z := \mathcal{S}^{n-1} \times \boldsymbol{R}$. If R is considered as an operator from $\mathcal{S}$ into $\mathcal{S}_e(Z)$, then its range is the subspace $\mathcal{S}_{em} \subset \mathcal{S}_e$ which consists precisely of the functions $h \in \mathcal{S}_e(Z)$ satisfying the moment conditions

$$\int_{-\infty}^{\infty} h(\alpha, p) p^k\,dp = \mathcal{P}_k(\alpha), \quad k = 0,1,2,\ldots, \tag{3}$$

where $\mathcal{P}_k(\alpha)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a restriction on $\mathcal{S}^{n-1}$ of a homogeneous polynomial of degree k defined on $\boldsymbol{R}^n$. The well known result is: $\mathcal{S}_{em} = R\mathcal{S}$.

Let $\langle f, \phi \rangle$ denote the $L^2(\boldsymbol{R}^n)$ inner product, and (g, h) denote the $L^2(Z)$ inner product. Let R^* be the adjoint operator

$$R^*h = \int_{S^{n-1}} h(\alpha, \alpha \cdot x)\,d\alpha. \tag{4}$$

2.2. The GG definition is based on the Parseval formula for the Radon transform:

$$\langle f, \phi \rangle = (\tilde{f}, \Phi) \tag{5}$$

where

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