Small Stable Stationary Solutions in Morrey Spaces of the Navier-Stokes Equation

By Hideo KOZONO*) and Masao YAMAZAKI**)

(Communicated by Kiyosi ITÔ, M. J. A., Nov. 13, 1995)

Recently, many authors studied the Cauchy problem for the Navier-Stokes equation in $\boldsymbol{R}^n$ in the framework of Morrey spaces. For example, Giga and Miyakawa [2] and Kato [3] gave sufficient conditions for the unique existence of time-global solutions. For previous papers related to this problem, see the references of Kozono and Yamazaki [4], which studied the above Cauchy problem in new function spaces larger than the corresponding Morrey spaces. However, these papers considered only the case where the external force vanishes identically or decays as $t \longrightarrow \infty$.

The purpose of this paper is to generalize the results on the global solvability in the works above to the case with a stationary external force by showing the unique existence and the stability of a small stationary solution in suitable Morrey spaces under appropriate assumptions on the external force.

More precisely, we consider the following stationary Navier-Stokes equation with an external force $f(x)$ in $\boldsymbol{R}^n$ for $n \geq 3$:

$$(1)\qquad -\Delta_x w(x) + (w(x) \cdot \nabla_x) w(x) + \nabla_x \pi(x) = f(x),$$

$$(2)\qquad \nabla_x \cdot w(x) = 0,$$

and find a sufficient condition on $f(x)$ for the unique existence of a small solution of (1)-(2) in suitable Morrey spaces.

We also verify the stability of the above stationary solution by showing the time-global unique solvability and giving a bound of the solution of the following nonstationary Navier-Stokes equation in $(0, \infty) \times \boldsymbol{R}^n_x$ with the same external force as above:

$$(3)\qquad \frac{\partial v}{\partial t}(t, x) - \Delta_x v(t, x) + (v(t, x) \cdot \nabla_x) v(t, x) + \nabla_x q(t, x) = f(x),$$

$$(4)\qquad \nabla_x \cdot v(t, x) = 0,$$

$$(5)\qquad v(0, x) = a(x) \text{ on } \boldsymbol{R}^n_x,$$

for the Cauchy data $a(x)$ close enough to the stationary solution.

Furthermore, we can take initial values in suitable function spaces introduced by [4]. These spaces are strictly larger than the corresponding Morrey spaces, and contain distributions other than Radon measures.

We start with the definition of the function spaces. Let p, q and s be real numbers such that $1 \leq q \leq p$, and suppose that $r \in [1, \infty]$. Then the *Morrey space* $\mathcal{M}_{p,q}$ on $\boldsymbol{R}^n$ is defined to be the set of functions $u(x) \in L^q_{\mathrm{loc}}(\boldsymbol{R}^n)$ such that

$$\| u \mid \mathcal{M}_{p,q} \| = \sup_{x_0 \in \boldsymbol{R}^n} \sup_{R>0} R^{n/p-n/q} \left(\int_{|x-x_0|<R} |u(x)|^q dx \right)^{1/q} < \infty.$$

We next define the space $\mathcal{M}^s_{p,q}$ by the formula

$$\mathcal{M}^s_{p,q} = \{ u(x) \in \mathcal{S}'/\mathcal{P} \mid \| u \mid \mathcal{M}^s_{p,q} \| = \| (-\Delta_x)^{s/2} u \mid \mathcal{M}_{p,q} \| < \infty \},$$

where $\mathcal{S}'$ and $\mathcal{P}$ denote the set of tempered distributions on $\boldsymbol{R}^n$ and the set of polynomials with n variables respectively.

Furthermore, we define the space $\mathcal{N}^s_{p,q,r}$ after [4] as the set of $u(x) \in \mathcal{S}'/\mathcal{P}$ such that

$$\| u \mid \mathcal{N}^s_{p,q,r} \| = \| \{ 2^{js} \| \mathcal{F}^{-1} [\varphi(2^{-j}\cdot) \mathcal{F}[u]] \mid \mathcal{M}_{p,q} \| \}^{\infty}_{j=-\infty} \mid \ell^r \| < \infty,$$

where $\{\varphi(2^{-j}\xi)\}^{\infty}_{j=-\infty}$ is a homogeneous Littlewood-Paley partition of unity. (See Bergh and Löfström [1] for example.)

Then it is shown in [4] that $\mathcal{N}^s_{p,q,1} \subset \mathcal{M}^s_{p,q} \subset \mathcal{N}^s_{p,q,\infty}$, and that the spaces $M^s_{p,q}$ and $\mathcal{N}^s_{p,q,r}$ can be canonically regarded as a subspace of $\mathcal{S}'$ if $s < n/p$.

Now we can state our main results.

Theorem A. *Suppose that* r *satisfies* $2 < r \leq n$. *Then there exist a positive number* δ_0 *and a continuous, strictly monotone-increasing function* $\omega(\delta)$ *on* $[0, \delta_0]$ *satisfying* $\omega(0) = 0$ *such that the following hold*:

(1) *For every* $f(x) \in (\mathcal{D}')^n$, *there exists at*

*) Graduate School of Mathematics, Nagoya University.

**) Department of Mathematics, Hitotsubashi University.