Duality for Hypergeometric Period Matrices

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We present some basic identities for the hypergeometric period matrices associated with the integrals of Euler type. Our main theorem shows not only identities classically known for integrals expressing hypergeometric series such as

$$(1)\qquad \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 t^a(1-t)^{c-a}(1-tx)^{-b}(1-ty)^{-b'}\frac{dt}{t(1-t)}$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(b')\Gamma(c-b-b')}\iint_{\substack{s>0,t>0\\1-s-t>0}} s^b t^{b'}(1-s-t)^{c-b-b'}(1-sx-ty)^{-a}\frac{ds\wedge dt}{st(1-s-t)}$$

but also identities for various hypergeometric functions. The full context of the theory will be published elsewhere.

Let $M(k+1, n+2)$ be the set of $(k+1)\times(n+2)$ complex matrices such that any $(k+1)$-minor does not vanish; for an element $x=(x_{ij})_{0\le i\le k, 0\le j\le n+1}\in M(k+1, n+2)$, put

$$L_j = L_j(t, x) = \sum_{i=0}^{k} t_i x_{ij},$$
$$H_j = H_j(x) = \{t\in \boldsymbol{P}^k \mid L_j(t, x) = 0\},$$
$$T(x) = \boldsymbol{P}^k - \bigcup_{j=0}^{n+1} H_j(x),$$
$$x\langle J\rangle = \det(x_{ij_m})_{0\le i,m\le k},$$

where $t=(t_0,\dots,t_k)$ is a homogeneous coordinate system of the complex projective space $\boldsymbol{P}^k$ and $J=\{j_0,\cdots,j_k\}$, $0\le j_0<j_1<\cdots<j_k\le n+1$ denotes a multi-index. We define a multi-valued function $U^\alpha = U^\alpha(t, x)$ and holomorphic k-forms $\varphi_J=\varphi_J(t, x)$ on $T(x)\times M(k+1, n+2)$ by

$$U^\alpha(t, x) = \prod_{j=0}^{n+1} L(t, x)^{\alpha_j}/\prod_J x\langle J\rangle^{(\alpha_{j_0}+\cdots+\alpha_{j_k})/\binom{n}{k}},$$
$$\varphi_J(t, x) = d_t\log(L_{j_0}(t, x)/L_{j_1}(t, x))\wedge\cdots\wedge d_t\log(L_{j_{k-1}}(t, x)/L_{j_k}(t, x)),$$

where

$$\alpha=(\alpha_0,\cdots,\alpha_{n+1}),\ \alpha_j\in \boldsymbol{C}\setminus\boldsymbol{Z},\ \textstyle\sum_{j=0}^{n+1}\alpha_j = 0.$$

Let ξ_k be a fixed element of $M(k+1, n+2)$ of the following form:

$$\xi_k = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0\\ \lambda_0 & \lambda_1 & \cdots & \lambda_n & 0\\ \lambda_0^2 & \lambda_1^2 & \cdots & \lambda_n^2 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ \lambda_0^k & \lambda_1^k & \cdots & \lambda_n^k & 1\end{pmatrix},$$
$$0\le\lambda_0<\lambda_1<\cdots<\lambda_n;$$

since $\xi_k\langle J\rangle$ is positive for any J, we assign the argument of $\xi_k\langle J\rangle$ by requiring $\arg(\xi_k\langle J\rangle)=0$. Let $\Delta_J=\Delta_J(\xi_k)$ be the simplex in $\boldsymbol{P}^k(\boldsymbol{R})\subset\boldsymbol{P}^k$ defined by the inequalities

$$(-1)^{k-m}(L_{j_m}(t, \xi_k)/L_{n+1}(t, \xi_k))>0,\quad j_m\in J;$$

we assign the argument of L_{j_m}/L_{n+1} on Δ_J by

$$\arg(L_{j_m}(t, \xi_k)/L_{n+1}(t, \xi_k)) = (k-m)\pi.$$

Note that $\Delta_J\cap H_j\neq\phi$ for $j_m<j<j_{m+1}$; we deform Δ_J to $\Delta_J^+=\Delta_J^+(\xi_k)\subset T(\xi_k)$ so that it is avoiding H_j, $j_m<j<j_{m+1}$ and that the arguments of L_j/L_{n+1} are assigned by

$$(k-m-1)\pi\le\arg(L_j(t, \xi_k)/L_{n+1}(t, \xi_k))\le(k-m)\pi,\ \text{for } j_m<j<j_{m+1}.$$

Let $\Delta_J^- = \Delta_J^-(\xi_k)$ be a deformation of Δ_J near H_j, $j\notin J$ on which the arguments of L_j/L_{n+1} are assigned by

$$\arg(L_{j_m}(t, \xi_k)/L_{n+1}(t, \xi_k)) \doteqdot -(k-m)\pi,\quad \text{for } j_m\in J$$
$$-(k-m)\pi\le\arg(L_j(t, \xi_k)/L_{n+1}(t, \xi_k))\le-(k-m-1)\pi,\ \text{for } j_m<j<j_{m+1};$$

see the following figure.

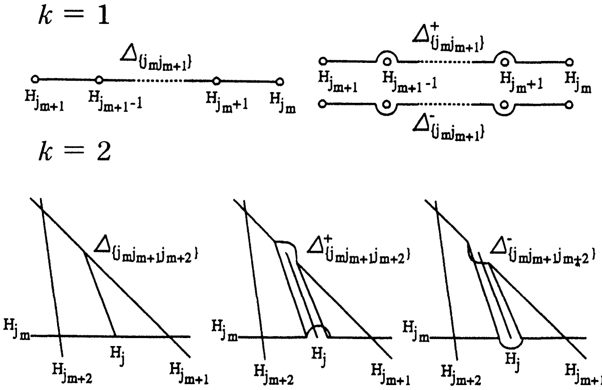

Fig.

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