A Lusternik-Schnirelmann Type Theorem for Locally Lipschitz Functionals with Applications to Multivalued Periodic Problems

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Abstract: We prove a Lusternik-Schnirelmann type theorem for locally Lipschitz functionals, by replacing the notion of Fréchet-differentiability with the Clarke generalized gradient. We apply our abstract framework to solve a multivalued second order periodic problem generated by non-smooth mappings.

Key words: Locally Lipschitz functional; Clarke subdifferential; Lusternik-Schnirelmann category; multivalued periodic problem.

1. Introduction. In the theory of differential equations two of the most important tools for proving the existence of solutions are the Mountain Pass Theorem of Ambrosetti-Rabinowitz and the Lusternik-Schnirelmann Theorem. These abstract results apply to the case where the solutions of the given problem are critical points of an appropriate functional of energy f, which is supposed to be real and of class C^1, defined on a real Banach space. The case when f fails to be differentiable arises frequently in non-smooth mechanics. In [8] we proved a generalization of the Mountain Pass Theorem for locally Lipschitz functionals. The aim of this paper is to give a variant of the Lusternik-Schnirelmann Theorem for such functionals.

We recall in what follows the main properties of locally Lipschitz functionals. For proofs and further details see [2] or [3].

Throughout, X will be a real Banach space. Let X^* be its dual and $\langle x^*, x\rangle$, for $x \in X$, $x \in X^*$, denote the duality pairing between X^* and X. Let $f : X \to \boldsymbol{R}$ be a locally Lipschitz ($f \in \mathrm{Lip}_{loc}(X, \boldsymbol{R})$). For each $x, v \in X$, we define the generalized directional derivative at x in the direction v of f as

$$f^0(x, v) = \limsup_{\substack{y \to x \\ \lambda \searrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda}$$

The generalized gradient (the Clarke subdifferential) of f at x is the subset $\partial f(x)$ of X^* defined by

$$\partial f(x) = \{x^* \in X^* ; f^0(x, v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\}$$

If f is convex, $\partial f(x)$ coincides with the subdifferential of f at x in the sense of convex analysis.

The fundamental properties of the Clarke subdifferential are:

a) For each $x \in X$, $\partial f(x)$ is a nonempty convex weak-★ compact subset of X^*.

b) For each $x, v \in X$, we have
$f^0(x, v) = \max\{\langle x^*, v\rangle ; x^* \in \partial f(x)\}$

c) The set-valued mapping $x \mapsto \partial f(x)$ is upper semi-continuous in the sense that for each $x_0 \in X$, $\varepsilon > 0$, $v \in X$, there is $\delta > 0$ such that for each $x^* \in \partial f(x)$ with $\| x - x_0 \| < \delta$, there exists $x_0^* \in \partial f(x_0)$ such that $| \langle x^* - x_0^*, v\rangle | < \varepsilon$.

d) The function $f^0(\cdot, \cdot)$ is upper semi-continuous.

e) If f achieves a local minimum or maximum at x, then $0 \in \partial f(x)$.

f) The function

$$\lambda(x) = \min_{x^* \in \partial f(x)} \| x^* \|$$

exists and is lower semi-continuous.

Definition 1. *A point $u \in X$ is said te be a critical point of $f \in \mathrm{Lip}_{loc}(X, \boldsymbol{R})$ if $0 \in \partial f(u)$, namely $f^0(u, v) \geq 0$ for every $v \in X$. A real number c is called a critical value of f if there is a critical point $u \in X$ such that $f(u) = c$.*

2. The main result. Let Z be a discrete subgroup of the real Banach space X, that is

$$\inf_{z \in Z \setminus \{0\}} \| z \| > 0$$

A function $f : X \to \boldsymbol{R}$ is said to be Z-*periodic* if $f(x + z) = f(x)$, for every $x \in X$ and