# Positive Solution of Some Nonlinear Elliptic Equation with Neumann Boundary Conditions*)

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**Abstract:** In this note we show that there exists $\Lambda_0$ such that, for every $\lambda \in (0, \Lambda_0)$, the problem: $-\Delta u = \lambda u^q + W(x)u^p$ in $\Omega$, $u > 0$ in $\Omega$, $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$, where $\Omega \subset R^N$ is a bounded convex domain with smooth boundary, $0 < q < 1 < p$ and $W \in C^1(\bar{\Omega})$, has a solution $u_\lambda$ iff $\int_\Omega W(x)dx < 0$. Moreover: $\| u_\lambda \|_\infty \to 0$ as $\lambda \downarrow 0$.

**1. Introduction.** In this note we study the Neumann problem for a class of semilinear elliptic equations.

Let $\Omega \subset R^N$ be a bounded convex domain with smooth boundary $\partial\Omega$ and consider the semilinear elliptic problem:

$$(1_\lambda) \begin{cases} -\Delta u = \lambda u^q + W(x)u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

where $0 < q < 1 < p$ and $W \in C^1(\bar{\Omega})$. The influence of negative part of $W$ is displayed in the following condition:

$$(*) \qquad \int_\Omega W(x)dx < 0.$$

As it turns out, condition $(*)$ was inspired by a corresponding necessary condition derived in [2]. The corresponding Dirichlet problem:

$$\begin{cases} -\Delta u = \lambda u^q + u^p & x \in \Omega \\ u > 0 & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

with $0 < q < 1 < p$, has been extensively studied in the paper of Ambrosetti, Brezis and Cerami [1]. Moreover, by the results of Boccardo, Escobedo and Peral [4], these results are extended for the p-laplacian. The purpose of the present note is to study $(1_\lambda)$ and our main result is the following:

**Theorem 1.1.** If $(*)$ is satisfied, then there exists$\Lambda_0 \in R$, $\Lambda_0 > 0$, such that, for all $\lambda \in (0, \Lambda_0)$, problem $(1_\lambda)$ has a solution $u_\lambda$ and $\| u_\lambda \|_\infty \to 0$ as $\lambda \downarrow 0$.

The proof of the above theorem uses only elementary tools. It is based on the construction of explicit sub and super solutions for $(1_\lambda)$ and the application of the Sattinger results (see [6]).

**2. The existence result.**

**Lemma 2.1.** Suppose there exists $\lambda > 0$ such that the problem $(1_\lambda)$ has a solution $u_\lambda$. Then necessarily the condition $(*)$ must hold.

*Proof.* For each $\varepsilon > 0$ put:

$$f_\varepsilon(u_\lambda) = \frac{1}{1-p}(u_\lambda + \varepsilon)^{1-p}.$$

We observe that:

$$-\Delta f_\varepsilon(u_\lambda) = (u_\lambda + \varepsilon)^{-p}(\lambda u_\lambda^q + W(x)u_\lambda^p) + p(u_\lambda + \varepsilon)^{-p-1}|\nabla u_\lambda|^2 \text{ in } \Omega,$$

$$\frac{\partial f_\varepsilon(u_\lambda)}{\partial n} = (u_\lambda + \varepsilon)^{-p}\frac{\partial u_\lambda}{\partial n} = 0 \qquad \text{on } \partial\Omega.$$

Hence:

$$-\int_\Omega W(x)\frac{u_\lambda^p}{(u_\lambda + \varepsilon)^p}dx = \int_\Omega p(u_\lambda + \varepsilon)^{-p-1}|\nabla u_\lambda|^2 dx + \lambda\int_\Omega \frac{u_\lambda^p}{(u_\lambda + \varepsilon)^p}dx.$$

It follows that there exists $\delta > 0$ such that:

$$\int_\Omega W(x)\frac{u_\lambda^p}{(u_\lambda + \varepsilon)^p}dx \leq -\delta < 0, \text{ for all } \varepsilon \in (0,1).$$

Letting $\varepsilon \to 0$, we have:

$$\int_\Omega W(x)dx \leq -\delta < 0.$$

Throughout, in the following, we suppose that the condition $(*)$ is satisfied.

**Lemma 2.2.** For all $\lambda > 0$, there exists a subsolution $\underline{u}_\lambda$, strictly positive in $\Omega$, for the problem $(1_\lambda)$.

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