Quadratic Relations between Logarithms of Algebraic Numbers

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So far, the four exponentials conjecture has been solved only in one special case, namely when the transcendence degree of the field which is spanned by the four logarithms is 1. We produce a new proof of this statement, and we announce a generalization: we replace the determinant of a 2×2 matrix by any homogeneous polynomial of degree 2.

§1. The results. The following statement provides a solution of the four exponentials conjecture in transcendence degree 1.

Theorem 1. *Let x_1 and x_2 be two complex numbers which are linearly independent over $\boldsymbol{Q}$, and similarly let y_1, y_2 be two $\boldsymbol{Q}$-linearly independent complex numbers. Assume that the field $\boldsymbol{Q}(x_1, x_2, y_1, y_2)$ has transcendence degree 1 over $\boldsymbol{Q}$. Then one at least of the four numbers*

$$e^{x_1y_1},\ e^{x_1y_2},\ e^{x_2y_1},\ e^{x_2y_2}$$

is transcendental.

For a proof of this result, we refer to [1] Cor. 7 and [6] Cor. 4.

Our is goal to introduce a sketch of new proof, where Gel'fond's transcendence criterion [3] (Chap. III, §4, lemma VII) is replaced by a diophantine approximation result due to Wirsing [8] §3. This new argument allows us to use Laurent's interpolations determinants [4] §6.

A generalization of theorem 1 can be achieved with the same arguments. Here we merely state the result; a complete proof (of a more general statement) will be given in another paper.

We denote by $\mathscr{L}$ the $\boldsymbol{Q}$-vector space of logarithms of algebraic numbers:

$$\mathscr{L} = \exp^{-1}(\bar{\boldsymbol{Q}}^{\times}) = \{z \in \boldsymbol{C}\,;\, e^z \in \bar{\boldsymbol{Q}}^{\times}\} \subset \boldsymbol{C},$$

where $\bar{\boldsymbol{Q}}$ is the algebraic closure of $\boldsymbol{Q}$ in $\mathbf{C}$ and $\bar{\boldsymbol{Q}}^{\times}$ is the multiplicative group of non-zero algebraic numbers.

Theorem 2. *Let $V \subseteq \mathbf{C}^n$ be the set of zeroes in $\boldsymbol{C}^n$ of a non-zero homogeneous polynomial $P \in \boldsymbol{Q}[X_1, \dots, X_n]$ of degree ≤ 2 and let $(\lambda_1, \dots, \lambda_n)$ be a point in V with coordinates in $\mathscr{L}$. Assume that the field $\boldsymbol{Q}(\lambda_1, \dots, \lambda_n)$ has transcendence degree 1 over $\boldsymbol{Q}$. Then $(\lambda_1, \dots, \lambda_n)$ is contained in a vector subspace of $\boldsymbol{C}^n$ which is defined over $\boldsymbol{Q}$ and contained in V.*

Theorem 1 is the special case of Theorem 2 when P is $X_1X_4 - X_2X_3$ with $n = 4$.

§2. Wirsing's theorem. When α is a complex algebraic number of degree $d = [\boldsymbol{Q}(\alpha) : \boldsymbol{Q}]$, we denote by $\mathrm{M}(\alpha)$ its Mahler's measure, which is related to its absolute logarithmic height $\mathrm{h}(\alpha)$ by

$$d\mathrm{h}(\alpha) = \log \mathrm{M}(\alpha).$$

The main tool of this paper is the following theorem of Wirsing [8]:

Theorem 3. *Let θ be a complex transcendental number. For any integer $D \geq 2$ there exist infinitely many algebraic numbers α which satisfy*

$$[\boldsymbol{Q}(\alpha) : \boldsymbol{Q}] \leq D \text{ and } |\theta - \alpha| \leq \mathrm{M}(\alpha)^{-D/4}.$$

§3. Laurent's interpolation determinants. A proof of the six exponentials theorem which does not rest on Dirichlet's box principle has been given by M. Laurent in [4]: he replaces the construction of an auxiliary function (which involves Thue-Siegel lemma) by an explicit determinant.

The following result is a variant of Proposition 9.5 of [7]: here, we include derivatives.

Proposition. *Let L be a positive integer, E and U be positive real numbers with $0 < \log E \leq 4U$. For $1 \leq \lambda \leq L$, let φ_λ be a complex function of one variable, $b_{\lambda 1}, \dots, b_{\lambda L}$ be complex numbers and M_λ a real number; further, for $1 \leq \mu \leq L$, let ζ_μ be a complex number and σ_μ be a non-negative integer. Assume that, for $1 \leq \lambda \leq L$, we have*

$$M_\lambda \geq \log \sup_{|z|=E} \max_{1\leq\mu\leq L} |((d/dz)^{\sigma_\mu}\varphi_\lambda)(z\zeta_\mu)|$$

$$\text{and } M_\lambda \geq \log \max_{1\leq\mu\leq L} |b_{\lambda\mu}|.$$

Finally, let ε be a complex number with $|\varepsilon| \leq e^{-U}$. Then the logarithm of the absolute value of the determinant

$$\Delta = \det((d/dz)^{\sigma_\mu}\varphi_\lambda(\zeta_\mu) + \varepsilon b_{\lambda\mu})_{1\leq\lambda,\mu\leq L}$$

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