Gamma Factors for Generalized Selberg Zeta Functions

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1. Introduction. Let K be an algebraic number field such that $[K:\mathbf{Q}] < \infty$, and $\zeta_K(s)$ be the Dedekind zeta function of K. The completed Dedekind zeta function $\widehat{\zeta_K}(s) = \zeta_K(s) \cdot \Gamma_K(s)$ has the symmetric functional equation: $\widehat{\zeta_K}(1-s) = \widehat{\zeta_K}(s)$. Here, the gamma factor is:

$$\Gamma_K(s) = |D_K|^{\frac{s}{2}} \Gamma_{\mathbf{R}}(s)^{r_1(K)} \Gamma_{\mathbf{C}}(s)^{r_2(K)},$$

where, D_K is the discriminant of K, $r_1(K)$ and $r_2(K)$ are the number of real and complex places of K respectively. We can consider $\Gamma_{\mathbf{R}}(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)$, $\Gamma_{\mathbf{C}}(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1)$ as a "basis" of gamma factors corresponding to infinite places.

In this article we consider **"gamma factors" for Selberg zeta functions.** (cf. Vignéras[6], Sarnak [5], Kurokawa[3]). We give a neat expression of "gamma factors" as in the case of Dedekind zeta functions. (Theorem 1) Furthermore, we obtain a simple proof of the functional equation of the Ruelle zeta function $R(s)$ for a compact $2n$-dimensional real hyperbolic space X (Theorem 2):

$$R(s) \cdot R(-s) = (-4\sin^2(\pi s))^{n \cdot (-1)^{n-1} vol(X)}.$$

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2. Selberg zeta functions. Let G be a connected semisimple Lie group of rank one with finite conter, K be a maximal compact subgroup of G. Let Γ be a co-compact torsion-free discrete subgroup of G. Then $X = \Gamma \backslash G/K$ is a compact locally symmetric space of rank one. For a given irreducible unitary representation τ of K, we denote by $Z_\tau(s)$ the Selberg zeta functions of X with K-type τ as is introduced by Wakayama [7].

For example, let X be a compact Riemann surface of genus $g \geq 2$. Then $X = \Gamma \backslash H$ where $H = SL(2, \mathbf{R})/SO(2)$ is the upper half plane, and Γ is the fundamental group $\pi_1(X)$ discretely embedded in $SL(2, \mathbf{R})$. For trivial τ, the Selberg zeta function $Z(s)$ of a compact Riemann surface is defined by the following Euler products:

$$Z(s) = \prod_{p \in P_\Gamma} \prod_{k=0}^{\infty} (1 - N(p)^{-(k+s)}).$$

Here P_Γ is the set of all primitive hyperbolic conjugacy classes, and the norm function $N(p) = \max\{|\text{ eigenvalues of } p|^2\}$. For other rank one Lie groups and non-trivial τ, $Z_\tau(s)$ is defined by similar but more complicated Euler products.

Selberg-Gangolli[2]-Wakayama[7] have shown that:

$Z_\tau(s)$ is meromorphic on $\mathbf{C}$, and tells informations about τ-spectrum:

$$\hat{G}_\tau = \{\pi \in \hat{G} \mid m_\Gamma(\pi) > 0, \pi|_K \ni \tau\},$$

where $m_\Gamma(\pi)$ is the multiplicity of a unitary representation π of G in the right regular representation π_Γ of G on $L^2(\Gamma \backslash G)$. (and in our case $m_\Gamma(\pi)$ is finite for all π.)

$Z_\tau(s)$ has moreover the functional equation:

$$(1) \quad Z_\tau(2\rho_0 - s) = \exp\left(\int_0^{s-\rho_0} \Delta_\tau(t)\,dt\right) Z_\tau(s).$$

where, $\rho_0 > 0$ is a constant depending only on G and $\Delta_\tau(t)$ is the "Plancherel" density with K-type τ, whose explicit formula is found in [7]. Hereafter we use **renormalized** ρ_0 and $\Delta_\tau(t)$ like as [4].

3. Gamma factors. we shall express the exponential factor of the functional equation (1) as $\Gamma_\tau(s)/\Gamma_\tau(2\rho_0 - s)$ by the "gamma factor" $\Gamma_\tau(s)$ so that the completed Selberg zeta function $\widehat{Z_\tau}(s) = Z_\tau(s)\Gamma_\tau(s)$ will satisfy the symmetric functional equation:

$$(2) \quad \widehat{Z_\tau}(2\rho_0 - s) = \widehat{Z_\tau}(s)$$

If $\dim X$ is odd, the "Plancherel" density $\Delta_\tau(t)$ is a polynomial and "gamma factor" is trivial. Hereafter we suppose that $\dim X$ is even, i.e. $G = SO(2n, 1), SU(n, 1), Sp(n, 1), F_4$. Then the "Plancherel" density is given by $\Delta_\tau(t) = \sum_{\text{finite sun}} (\text{odd polynomial})\pi(\tan(\pi t))^{\pm 1}$.

Definition 3.1. *We define two "Plancherel polynomials" $P_\tau(t)$ and $Q_\tau(t)$ attached to τ by,*

$$(-1)^{\dim X/2} vol(X)^{-1} \Delta_\tau(t) = -P_\tau(t)\pi\cot(\pi t) + Q_\tau(t)\pi\tan(\pi t).$$

These polynomials are odd polynomials of degree