A Construction of Exceptional Simple Graded Lie Algebras of the Second Kind

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§0. Introduction. Let $\mathfrak{g} = \sum_{k=-2}^{2} g_k$ be a *graded Lie algebra of the second kind* (shortly 2-GLA). In [5], Kaneyuki gave the classification of exceptional real simple 2-GLA's and listed up the subalgebras g_0 and the dimension of $g_k (k = 1, 2)$. Since the subspaces $g_k (k \neq 0)$ were not explicitly determined in [5], we will give an explicit representation of g_k in this paper. Up to the present, several constructions of 2-GLA have been thought out. Allison ([1]) gave a construction of 2-GLA starting from structurable algebra. His construction is useful but some exceptional real simple 2-GLA's can not be obtained by his construction. Details and proofs will be found in [3].

§1. Methods of construction. In this section, we give two methods of construction of 2-GLA.

1.1 Let g_0 be a real Lie algebra and $V_k (k = 1, 2)$ a real vector space with a nondegenerate symmetric bilinear form (,). For each element $\boldsymbol{u}$ of V_k, the element $\boldsymbol{u}^*$ of the dual space V_k^* is defined by $\boldsymbol{u}^*(\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v})$ $(\boldsymbol{v} \in V_k)$. Let ρ_k be a representation of g_0 on $V_k (k = 1, 2)$. By ρ_k^*, we denote the dual representation of ρ_k, that is

$$(\rho_k^*(X)\boldsymbol{u}^*)(\boldsymbol{v}) + \boldsymbol{u}^*(\rho_k(X)\boldsymbol{v}) = 0 \quad (\boldsymbol{u}, \boldsymbol{v} \in V_k, X \in g_0).$$

Now, we assume that the following bilinear maps are given.

$$\Delta : V_2 \times V_1^* \to V_1, \quad \circ : V_1 \times V_1 \to V_2 \ \text{(antisymmetric)},$$
$$\times : V_1 \times V_1^* \to g_0, \quad * : V_2 \times V_2^* \to g_0.$$

Let us consider the real vector space

$$\mathfrak{g} = g_0 \oplus V_1 \oplus V_1^* \oplus V_2 \oplus V_2^*.$$

We define a bilinear bracket operation in $\mathfrak{g}$ as follows:

$$(X, \boldsymbol{u}, \boldsymbol{v}^*, \boldsymbol{x}, \boldsymbol{y}^*) = [(X_1, \boldsymbol{u}_1, \boldsymbol{v}_1^*, \boldsymbol{x}_1, \boldsymbol{y}_1^*), (X_2, \boldsymbol{u}_2, \boldsymbol{v}_2^*, \boldsymbol{x}_2, \boldsymbol{y}_2^*)],$$

where

$$\begin{cases} X = [X_1, X_2] + \boldsymbol{u}_1 \times \boldsymbol{v}_2^* - \boldsymbol{u}_2 \times \boldsymbol{v}_1^* + \boldsymbol{x}_1 * \boldsymbol{y}_2^* - \boldsymbol{x}_2 * \boldsymbol{y}_1^*, \\ \boldsymbol{u} = \rho_1(X_1)\boldsymbol{u}_2 - \rho_1(X_2)\boldsymbol{u}_1 + \boldsymbol{x}_1 \Delta \boldsymbol{v}_2^* - \boldsymbol{x}_2 \Delta \boldsymbol{v}_1^*, \\ \boldsymbol{v}^* = \rho_1^*(X_1)\boldsymbol{v}_2^* - \rho_1^*(X_2)\boldsymbol{v}_1^* - (\boldsymbol{y}_1 \Delta \boldsymbol{u}_2^*)^* + (\boldsymbol{y}_2 \Delta \boldsymbol{u}_1^*)^*, \\ \boldsymbol{x} = \rho_2(X_1)\boldsymbol{x}_2 - \rho_2(X_2)\boldsymbol{x}_1 + \boldsymbol{u}_1 \circ \boldsymbol{u}_2, \\ \boldsymbol{y}^* = \rho_2^*(X_1)\boldsymbol{y}_2^* - \rho_2^*(X_2)\boldsymbol{y}_1^* - (\boldsymbol{v}_1 \circ \boldsymbol{v}_2)^*. \end{cases}$$

In [3], we give a necessary and sufficient condition for $\mathfrak{g}$ to be a Lie algebra. When $\mathfrak{g}$ is a Lie algebra, obviously $\mathfrak{g} = \sum_{k=-2}^{2} g_k (g_k = V_k, g_{-k} = V_k^*)$ becomes a 2-GLA.

1.2. Let $\mathfrak{g} = \sum_{k=-2}^{2} g_k$ be a 2-GLA and γ a grade-preserving involution (= involutive automorphism) of $\mathfrak{g}$. Put

$$\mathfrak{g}_\gamma = \{X \in \mathfrak{g} \mid \gamma(X) = X\},$$
$$(g_k)_\gamma = \{X \in g_k \mid \gamma(X) = X\}.$$

If $(g_{\pm 2})_\gamma \neq (0)$, then the subalgebra $\mathfrak{g}_\gamma = \sum_{k=-2}^{2} (g_k)_\gamma$ also becomes a 2-GLA.

§2. The main theorem. Using g_0 and dim g_k listed up in [5], we construct the corresponding 2-GLA's by the methods described in §1. Then we have the following theorem.

Theorem 1. *The exceptional real simple graded Lie algebras of the second kind are realized as listed in Table* I.

In Table I, we use the following notations.

$\boldsymbol{C}$ (resp. $\boldsymbol{C}'$): the algebra of complex (resp. split complex) numbers

$\boldsymbol{H}$ (resp. $\boldsymbol{H}'$): the algebra of quaternion (resp. split quaternion) numbers

$\mathfrak{C}$ (resp. $\mathfrak{C}'$): the division Cayley (resp. split Cayley) algebra

For a real vector space V, its complexification $\{\boldsymbol{u} + i\boldsymbol{v} \mid \boldsymbol{u}, \boldsymbol{v} \in V\}$ is denoted by V^C. We do not identify $\boldsymbol{R}^C$ with $\boldsymbol{C}$, but denote $\boldsymbol{R}^C$ by C.

From now on, we explain the contents of Table I.

(1) In case of (e1)~(e9) and (e24)~(e27):