Extremal Kähler Metrics and the Calabi Energy*)

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Let (M, Ω) be a compact complex manifold with a distinguished Kähler class. By abuse of terminology, we say a Kähler metric g is in Ω if the Kähler form ω of g is in Ω. The volume and total scalar curvature of a Kähler metric in Ω depend only on Ω, and are denoted V_Ω and S_Ω respectively. The *Calabi energy* of a metric is defined to be

$$\Phi_\Omega(g) = \int_M s_g^2 \,\mathrm{dvol}_g, \tag{1}$$

and is obviously bounded below by S_Ω^2/V_Ω. A critical metric for the Calabi energy-among metrics representing the same Kähler class-is called an *extremal* Kähler metric.

Calabi ([3], page 99) asked whether or not the functional Φ_Ω has a unique critical value, and if so, whether or not extremal metrics are in fact global minima of the energy in their Kähler class. The main result of this note is to announce that the answer to both questions is "Yes." Moreover, the critical energy can be determined *a priori,* without reference to an extremal metric. (This was proven independently by Simanca [8].) The proof relies on three ingredients: The Euler equation for critical metrics of the Calabi energy; the Futaki character; and a natural complex-bilinear pairing, due to Futaki and Mabuchi [4], on the space of holomorphic gradient vector fields. We state the precise result, outline the proof, then suggest some possible consequences.

Let $\mathscr{F}_\Omega$ denote the Futaki character of Ω, and let X_Ω denote an extremal Kähler vector field, see Futaki and Mabuchi [4]. Intuitively, the vector field X_Ω is dual to the character $\mathscr{F}_\Omega$ under a canonical complex-bilinear pairing on the space of holomorphic gradient vector fields, but because this pairing is degenerate, the vector field X_Ω is not well-defined. However, the value $\mathscr{F}_\Omega(X_\Omega)$ *is* well-defined (see [4]), and is a nonnegative real number (see [5]).

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Theorem A. *For each metric g with fundamental class Ω,*

$$\Phi_\Omega(g) \geq S_\Omega^2/V_\Omega + \mathscr{F}_\Omega(X_\Omega), \tag{2}$$

with equality if and only if g is a critical metric for Φ_Ω.

This result highlights the close relationship between the Futaki character and the Calabi energy, and has an amusing statistical interpretation. The quantity $\Phi_\Omega(g) - S_\Omega^2/V_\Omega$ is the variance of the scalar curvature function s_g, computed on the measure space (M, dvol_g). Theorem A says that this variance is *a priori* bounded below by $\mathscr{F}_\Omega(X_\Omega)$ as the metric g ranges in Ω. In other words, equation (2) asserts that the norm squared of the Futaki character is a precise numerical measure of how far an extremal metric is from having constant scalar curvature. This nicely encodes both the result of Futaki that a Kähler class containing a metric with constant scalar curvature must have vanishing Futaki character, and the result of Calabi that if the Futaki character $\mathscr{F}_\Omega$ vanishes, then an extremal metric in Ω has constant scalar curvature.

The idea of the proof is quite simple. Fix any Kähler metric g whose Kähler form ω represents the class Ω. There is a finite-dimensional space of smooth, complex-valued functions, called *g-holomorphy potentials,* whose gradients with respect to g are holomorphic. Write the scalar curvature function s of g as an L_g^2-orthogonal sum

$$s = \mathbf{H}s + \Pi s + s^\perp, \tag{3}$$

where $\mathbf{H}s = S_\Omega/V_\Omega$ is the average scalar curvature (i.e. the g-harmonic part), and $\mathbf{H}s + \Pi s$ is the g-orthogonal projection of s into the space of g-holomorphy potentials. (*n.b.* The projection Π here is written $\Pi_g - \mathbf{H}$ in [5].) Since M is compact, the metric g is extremal if and only if the gradient field of the scalar curvature is holomorphic (see for example, [2]). This is the same as saying the scalar curvature s is a holomorphy potential, or that $s^\perp = 0$.