A Decomposition of R-polynomials and Kazhdan-Lusztig Polynomials

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The R-polynomial is defined for two elements in an arbitrary Coxeter group. These polynomials are intimately related to Kazhdan-Lusztig polynomials introduced by Kazhdan and Lusztig in 1979 ([4]). For example, it is well known that

$$q^{l(w)-l(x)}P_{x,w}\left(\frac{1}{q}\right) = \sum_{x \leq y \leq w} R_{x,y}(q)P_{y,w}(q),$$

where $P_{x,w}(q)$ (resp. $R_{x,w}(q)$) is the Kazhdan-Lusztig polynomial (resp. the R-polynomial).

In [2], F. Brenti found a decomposition formula of R-polynomials for symmetric groups and he showed that products of R-polynomials for symmetric groups are also R-polynomials for symmetric groups. The purpose of this article is to find a decomposition formula of R-polynomials and Kazhdan-Lusztig polynomials for arbitrary Coxeter groups in extension of Brenti's result.

First, we recall the definition of the Bruhat order and R-polynomials. Throughout this article, (W, S) is an arbitrary Coxeter system, where S denotes a privileged set of involutions in W. The standard references are [1] and [3] for the Bruhat order and R-polynomials.

Definition (Bruhat order). We put $T := \{wsw^{-1}; s \in S, w \in W\}$. For $y, z \in W$, we denote $y <' z$ if and only if there exists an element t of T such that $l(tz) < l(z)$ and $y = tz$, where l is the length function. Then the Bruhat order denoted by $\leq$ is defined as follows. For x, $w \in W$, $x \leq w$ if and only if there exists a sequence $x_0, x_1, \ldots, x_r$ in W such that $x = x_0 <' x_1 <' \cdots <' x_r = w$.

The following is well known. For $w \in W$, let $s_1 s_2 \cdots s_m$ be a reduced expression of w, i.e. $w = s_1 s_2 \cdots s_m$, $s_i \in S$ for all $i \in [m] (:= \{1,2,\ldots, m\})$ and $l(w) = m$. For $x \in W$, $x \leq w$ if and only if there exists a sequence of natural numbers $i_1, i_2, \ldots, i_t$ such that $1 \leq i_1 < i_2 < \cdots < i_t \leq m$ and $x = s_{i_1} s_{i_2} \cdots s_{i_t}$. This expression of x is not reduced in general, i.e. it may happen that $l(x) < t$. However it is known that one can find a sequence of natural numbers $j_1, j_2, \ldots, j_k$ such that $1 \leq j_1 < j_2 < \cdots < j_k \leq m$, $x = s_{j_1} s_{j_2} \cdots s_{j_k}$ and $l(x) = k$.

Also, the following decomposition called the coset decomposition is well known. Let J be a subset of S. We put $W_J :=$ subgroup of W generated by J and $W^J := \{y \in W ; l(yz) = l(y) + l(z)$ for any $z \in W_J\}$. Then, for $w \in W$, there uniquely exist $w^J \in W^J$ and $w_J \in W_J$ such that $w = w^J w_J$, whence follows:

Lemma A. *Let $y, z \in W$. If $G(y) \cap G(z) = \phi$, where $G(y) := \{s \in S ; s \leq y\}$, then we have $l(yz) = l(y) + l(z)$.*

R-polynomials are defined as follows:

Definition–Proposition (R-polynomial). *$\mathcal{H}(W)$ is the Hecke algebra associated to W. That is, $\mathcal{H}(W)$ is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w ; w \in W\}$ as a basis with the multiplication such that*

$$T_w T_s = \begin{cases} T_{ws} & \text{if } l(ws) > l(w), \\ qT_{ws} + (q-1)T_w & \text{if } l(ws) < l(w) \end{cases}$$

for all $w \in W$ and $s \in S$. For $w \in W$, there exists a unique family of polynomials $\{R_{x,w}(q)\}_{x \leq w} \subset \mathbf{Z}[q]$ satisfying

$$(T_{w^{-1}})^{-1} = q^{-l(w)} \sum_{x \leq w} (-1)^{l(w)-l(x)} R_{x,w}(q) T_x.$$

We put $R_{x,w}(q) := 0$ if $x \not\leq w$ for convenience. $R_{x,w}(q)$ is called the R-polynomial for x, $w \in W$.

By using R-polynomials, we can define Kazhdan-Lusztig polynomials as follows:

Definition–Proposition (Kazhdan-Lusztig polynomial). *There exists a unique family of polynomials $\{P_{x,w}(q)\}_{x,w \in W} \subset \mathbf{Z}[q]$ satisfying the following conditions:*

(i) $P_{x,w}(q) = 0$ *if* $x \not\leq w$,

(ii) $P_{x,x}(q) = 1$,

(iii) $\deg P_{x,w}(q) \leq \frac{1}{2}(l(w) - l(x) - 1)$ *if* $x < w$,

(iv) $q^{l(w)-l(x)}P_{x,w}\left(\frac{1}{q}\right) = \sum_{x \leq y \leq w} R_{x,y}(q)P_{y,w}(q)$ *if* $x \leq w$.

$P_{x,w}(q)$ is called the Kazhdan-Lusztig polynomial