17. *Coefficient Bounds for the Inverse of a Function whose Derivative has a Positive Real Part*

By CHOU Shiqiong

Department of Mathematics, Changsha Communications Institute, People's Republic of China

(Communicated by Kiyosi ITÔ, M. J. A., March 14, 1994)

Abstract: In this paper we study the coefficient bounds for the inverse of a function whose derivative has a positive real part. We prove the conjecture posed by R. J. Libera and E. J. Złotkiewicz [3].

1. Introduction and conclusion. Let S denote the class of functions of the form

(1) $$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic and univalent in $\Delta = \{z : |z| < 1\}$. De Branges [1] has proved that $a_n (n = 2,3,\cdots)$ are bounded by those of the Koebe function, $k(z) = z + 2z^2 + 3z^3 + \cdots$, that is, $|a_n| \leqslant n \ (n \geqslant 2)$.

The inverse of $f(z)$ has a series expansion in some disk about the origin of the form

(2) $$F(w) = w + \gamma_2 w^2 + \gamma_3 w^3 + \cdots.$$

It was shown early (see [2]) that the inverse of the Koebe function provides the best bound for all $|\gamma_k|$.

As is usually the case, we let $\mathscr{P}$ be the family of functions

(3) $$p(z) = 1 + c_1 z + c_2 z^2 + \cdots$$

regular and with $\operatorname{Re} p(z) > 0 (z \in \Delta)$. Furthermore we denote by J the class of all functions of form (1) which satisfies

(4) $$\operatorname{Re} f'(z) > 0, \ z \in \Delta.$$

This is the family studied widely. Let the inverse of $f(z)$ belonging to J have the form (2). R. J. Libera and E. J. Złotkiewicz [3] found sharp bounds for the first six coefficients of $F(w)$; the extremal function is $\tilde{F}_0(w)$ which corresponds to $\tilde{f}_0(z) = -z - 2\log(1-z)$. They also conjectured that $\tilde{F}_0(w)$ gives the sharp upper bounds for other (perhaps even all) coefficients. In this paper we prove this conjecture, and the method is very succinct. Our conclusion is

Theorem. Let $f(z)$ be in J and the inverse of $f(z)$ be $F(w) = w + \sum_{n=2}^{\infty} \gamma_n w^n$. Then

$$|\gamma_n| \leqslant B_n (n = 2,3,\cdots)$$

where $B_n (n = 2,3,\cdots)$ are the coefficients of $F_0(w)$ which corresponds to $f_0(z) = -z + 2\log(1+z)$. The function attaining the equalities is the inverse of $f_0(z)$.

2. The proof of the theorem. It's easy to know that $1/p(z) \in \mathscr{P}$ when $p(z) \in \mathscr{P}$. So if $f(z) \in J$, then there exists a $p(z) \in \mathscr{P}$ such that