14. Determination of All Quaternion Octic CM-fields with Ideal Class Groups of Exponents 2 Abridged Version

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 14, 1994)

In [9] the authors set to determine the non-abelian normal CM-fields with class number one. Since they have even relative class numbers, they got rid of quaternion octic CM-fields. Here, a quaternion octic field is a normal number fields of degree 8 whose Galois group is the quaternion group $\mathbf{G} = \{\pm 1, \pm i, \pm j, \pm k\}$ with $ij = k, jk = i, ki = j$ and $i^2 = j^2 = k^2 = -1$. Then, in [8] the first author determined the only quaternion octic CM-fields with class number 2. Here, we delineate the proof of the following result proved in [10] that generalizes this previous result:

Theorem. *There are exactly* 2 *quaternion octic* CM-*fields with ideals class groups of exponents* 2. *Namely, the following two pure quaternion number fields*:

$$Q(\sqrt{-(2+\sqrt{2})(3+\sqrt{6})})$$

with discriminant $2^{24}3^6$ *and class number* 2, *and*

$$Q(\sqrt{-(5+\sqrt{5})(5+\sqrt{21})(21+\sqrt{105})})$$

with discriminant $3^6 5^6 7^6$ *and class number* 8.

1. Analytic lower bounds for relative class numbers and maximal real subfields of quaternion octic CM-fields with ideal class groups of exponents 2. Here we show that under the assumption of a suitable hypothesis (H) we can set lower bounds on relative class numbers of quaternion octic CM-fields. Let us remind the reader that a number field N is called a CM-field if it is a totally imaginary number field that is a quadratic extension of a totally real subfield K. In that situation, one can prove that the class number h_K of K divides that h_N of N, and the relative class number h_N^- of N is defined by means of $h_N^- = h_N / h_K$ (see [11, Theorem 4.10]). Note h_N^- divides h_N.

Proposition 1. (a). (See [5, Theorems 1 and 2(a)]) *Let N be a quaternion octic* CM-*field such that the Dedekind zeta function of its real bicyclic biquadratic subfield K satisfies*

$$\text{(H)} \qquad \zeta_K\left(1 - \frac{2}{\log(D_N)}\right) \le 0.$$

Then, we have the following lower bound for the relative class number h_N^- of N:

$$\text{(1)} \qquad h_N^- \ge \left(1 - \frac{8\pi e^{1/4}}{D_N^{1/8}}\right) \frac{1}{4e\pi^4} \frac{1}{\mathrm{Res}_{s=1}(\zeta_K)} \frac{\sqrt{D_N/D_K}}{\log(D_N)}.$$

Moreover, the hypothesis (H) *is satisfied provided that we have*

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