11. *Some Families of Generalized Hypergeometric Functions Associated with the Hardy Space of Analytic Functions*

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Abstract: Recently, several inclusion theorems associated with the Hardy space of analytic functions were proven for various families of generalized hypergeometric functions belonging to one or the other subclasses of the class $\mathcal{A}$ of normalized analytic functions in the open unit disk $\mathcal{U}$. The main objective of this paper is to develop a remarkably simple proof of a unification (and generalization) of many of these inclusion theorems. Some relevant historical remarks and observations are also presented.

1. Introduction and definitions. Let $\mathcal{A}$ denote the class of functions $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are *analytic* in the *open* unit disk $\mathcal{U}$. Also let $\mathcal{S}$ denote the class of all functions in $\mathcal{A}$ which are *univalent* in $\mathcal{U}$. We denote by $\mathcal{S}^*$ and $\mathcal{K}$ the subclasses of $\mathcal{S}$ consisting of all functions in $\mathcal{A}$ which are, respectively, *starlike* and *convex* in $\mathcal{U}$. Then it follows readily that $f(z) \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}$, which indeed is the familiar Alexander theorem (cf., e.g., Duren [3, p.43, Theorem 2.12]). We note also that $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$.

Let $\mathcal{H}^p (0 < p \leq \infty)$ denote the Hardy space of analytic functions $f(z)$ in $\mathcal{U}$, and define the integral means $M_p(r, f)$ by

$$M_p(r, f) = \begin{cases} \left(\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta\right)^{1/p} & (0 < p < \infty) \\ \sup\limits_{0 \leq \theta < 2\pi} |f(re^{i\theta})| & (p = \infty). \end{cases} \tag{1.2}$$

Definition 1. A function $f(z)$, analytic in $\mathcal{U}$, is said to belong to the Hardy space $\mathcal{H}^p (0 < p \leq \infty)$ if

$$\lim_{r \to 1-} \{M_p(r, f)\} < \infty \quad (0 < p \leq \infty). \tag{1.3}$$

For $1 \leq p \leq \infty$, $\mathcal{H}^p$ is a Banach space with the norm $\|f\|_p$ defined by (cf., e.g., Duren [2, p. 23]; see also Koosis [11])

$$\|f\|_p = \lim_{r \to 1-} \{M_p(r, f)\} \quad (1 \leq p \leq \infty). \tag{1.4}$$

Furthermore, $\mathcal{H}^\infty$ is the familiar class of *bounded* analytic functions in $\mathcal{U}$,

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