60. *An Application of Frey's Idea to Exponential Diophantine Equations*

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(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1994)

In 1956 Sierpiński ([6]) proved that the equation $3^x + 4^y = 5^z$ has no solutions in natural numbers x, y, z except one, namely, $x = y = z = 2$. Jeśmanowicz ([5]) soon followed him by proving that the only solution of each of the equations

$$5^x + 12^y = 13^z,\ 7^x + 24^y = 25^z,\ 9^x + 40^y = 41^z,\ 11^x + 60^y = 61^z$$

in natural numbers x, y, z is also $x = y = z = 2$, and asked whether there exists a Pythagorean triple (a, b, c) for which the equation $a^x + b^y = c^z$ has a solution in natural numbers x, y, z different from $x = y = z = 2$ (cf. also [8]-[12]).

In the present paper, applying Frey's idea ([3]) (which reduces Fermat's problem to the existence of a certain kind of elliptic curve) to Jeśmanowicz's problem, we will give an effective procedure to determine whether there are no other solutions than $x = y = z = 2$ for a given Pythagorean triple.

This method, in fact, has much broader application as follows:

Theorem. *Let a, b, c, l, m, n be given relatively prime natural numbers. Then the equation*

$$la^x + mb^y = nc^z \tag{1}$$

has finitely many solutions, all of which can be effectively determined.

In what follows we will give a proof of this theorem and a few examples.

The following theorem is useful in reducing calculation:

Frey's theorem ([3]). *For relatively prime integers a, b, c and natural numbers n_1, n_2, n_3 suppose that (n_1, n_2, n_3) is a solution of the equation* (1) *with $la^{n_1} \equiv \pm 1 \pmod 4$ and $mb^{n_2} \equiv 0 \pmod{2^4}$. Let E be the elliptic curve defined by the equation*

$$y^2 = x(x - la^{n_1})(x + mb^{n_2}). \tag{2}$$

Then the curve E is stable with discriminant

$$(lmna^{n_1}b^{n_2}c^{n_3}/16)^2$$

and conductor

$$\prod_{p \mid lmnab^{n_2}c/16} p.$$

Now, let (a, b, c) be a primitive Pythagorean triple with b even (and hence divisible by 4), and suppose that there exist natural numbers n_1, n_2, n_3 which satisfy the equation

$$a^{n_1} + b^{n_2} = c^{n_3}. \tag{3}$$

Next, define the elliptic curve E by the equation

$$y^2 = x(x - a^{n_1})(x + b^{n_2}).$$

The discriminant Δ and the conductor N of the curve E are