58. *Selfsimilar Shrinking Curves for Anisotropic Curvature Flow Equations*

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We consider a simple looking ordinary differential equation of the form

$$u'' + u - \frac{a(\theta)}{u} = 0 \text{ in } \boldsymbol{R} \tag{1}$$

with a given positive function $a(\theta)$. This equation arises in describing selfsimilar solutions of anisotropic curvature flow equations. Since θ is the argument of the normal $\vec{n}$ of the curve, it is natural to impose 2π-periodicity for $a(\theta)$ in (1) and to ask for existence and uniqueness of 2π-periodic solutions.

The physical background of the above problem is an evolution equation for embedded closed curves $\{\Gamma_t\}_{t>0}$ in $\boldsymbol{R}^2$ (see [10]):

Consider an equation for Γ_t, where the normal velocity V is given by the curvature k weighted by a direction-dependent factor $a(\theta)$, i.e.

$$V = a(\theta)k, \quad a(\theta) = \beta(\theta)^{-1}(\gamma''(\theta) + \gamma(\theta)),$$

where β and $\gamma'' + \gamma$ are assumed to be positive, so that the equation is parabolic. γ is called the surface energy density and β is called the kinetic coefficient.

In case $a(\theta) \equiv$ const. it is well known (see [3], [4], [6] and [9]) that any initial curve becomes convex, after this it extincts in finite time, and that the type of shrinking is asymptotically similar to that of a shrinking circle $C_t = (2(t_* - t))^{1/2} C$, where C denotes the unit circle centered at the origin. (Here the time t_* is the extinction time and λC denotes the dilation of C with multiplier λ.) The curvature of the circle then is a solution of (1).

In case of more general $a(\theta)$, it was shown in [12] that selfsimilar solutions, i.e. solutions satisfying

$$\Gamma_t = (2(t_* - t))^{1/2}\Gamma$$

and thereby equation (1), exist if $\beta(\theta)\gamma(\theta) =$ const. Then Γ defined as the boundary of the so-called Wulff-Shape W_γ, i.e.

$$W_\gamma := \{x \in \boldsymbol{R}^2 \mid x\cdot\vec{m}(\sigma) \leq \gamma(\sigma) \text{ for all } \sigma \in \boldsymbol{R}\}, \tag{2}$$

yields a solution Γ_t of the evolution problem. Here $\vec{m}(\sigma)$ denotes a unit vector whose argument equals σ.

Our existence result now shows that such selfsimilar solutions exist for arbitrary positive $a(\theta)$. To simplify the notation we notice that a 2π-periodic function can be regarded as a function on the flat torus $\boldsymbol{T} := \boldsymbol{R}/2\pi\boldsymbol{Z}$. Thus we define

$$C^2_+(\boldsymbol{T}) = \{u \in C^2(\boldsymbol{R}) \mid u(\theta + 2\pi) = u(\theta) \text{ for all } \theta \in \boldsymbol{R},\ u > 0\}.$$

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