56. *The Central Limit Theorem for Rademacher System*

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1. Introduction. Let (Ω, P) be the Lebesgue probability space, i.e., $\Omega := [0,1)$ and P is Lebesgue measure. In this note, we regard any function on Ω as a function on $\boldsymbol{R}$ with period 1. The Rademacher system is a system $\{r_i\}$ of random variables on (Ω, P) defined by

$$r_1(\omega) := -1_{[0,1/2)}(\omega) + 1_{[1/2,1)}(\omega) \text{ and } r_i(\omega) := r_1(2^{i-1}\omega), \quad (i \geq 2).$$

Note that $(r_i + 1)/2$ gives the i-th digit of the dyadic expansion of a real number ω. Since $\{r_i\}$ is i.i.d., the De Moivre-Laplace theorem claims that the law of

$$X^{(m)}(\omega) := \frac{1}{\sqrt{m}} \sum_{i=1}^{m} r_i(\omega)$$

converges weakly to the standard normal distribution, as $m \to \infty$.

We put $X_n^{(m,\alpha)}(\omega) := X^{(m)}(\omega + n\alpha)$ for $n \in \boldsymbol{Z}$ and $\alpha \in \boldsymbol{R}$, and study the limit behaviour of the sequence $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ as $m \to \infty$. Since this sequence is given by iterating the Weyl automorphism, it is stationary and, in most cases, dependent. Having studied the quasi-Monte Carlo method, Sugita [6] conjectured that the dependence disappears as $m \to \infty$, for almost all α. He proved that, for almost all α,

$$R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)}) = o(m^{-\beta}) \text{ as } m \to \infty, \ (k \in \boldsymbol{N},\ 0 < \beta < 1/2).$$

We prove the following results related to the conjecture.

Theorem 1. *For almost all α with respect to Lebesgue measure, any finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges weakly to the multi-dimensional standard normal law as $m \to \infty$; i.e., for all $n \in \boldsymbol{Z}$ and $k \in \boldsymbol{N}$,*

$$(1.1) \qquad (X_n^{(m,\alpha)}, \dots, X_{n+k-1}^{(m,\alpha)}) \xrightarrow{\mathcal{D}} N(0, I_k) \text{ as } m \to \infty.$$

Here $\xrightarrow{\mathcal{D}}$ denotes convergence in law, and I_k the k-dimensional unit matrix.

Theorem 2. *For any α, the correlation $R^{(m,\alpha)}(k) := E(X_n^{(m,\alpha)} X_{n+k}^{(m,\alpha)})$ is given by*

$$(1.2) \quad R^{(m,\alpha)}(k) = \frac{1}{m} \sum_{i=1}^{m} \varphi(2^{i-1}k\alpha), \text{ where } \varphi(x) := |4x - 2| - 1, \ (x \in \Omega).$$

Moreover, for any $k \in \boldsymbol{N}$ and for almost all $\alpha \in \boldsymbol{R}$ with respect to Lebesgue measure, it holds that

$$(1.3) \qquad \limsup_{m\to\infty} \sqrt{\frac{m}{\log\log m}}\, R^{(m,\alpha)}(k) = \sqrt{\frac{2}{3}}.$$

Theorem 3. *The Hausdorff dimension of the set of α for which* (1.1) *does not hold is* 1.

Remark. We can improve Theorem 3 as follows: The Hausdorff dimension of the set of α such that finite dimensional distribution of $\{X_n^{(m,\alpha)}\}_{n\in\boldsymbol{Z}}$ converges to that of some stationary dependent gaussian sequence is 1. The