54. *On the Logarithmic Gradient of Poincaré Metric*

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1. **Introduction.** Let $\mathscr{D} \subset \mathscr{C}$ be a simply connected domain with at least two boundary points and let $f(z)$ be a conformal mapping of $\mathscr{B} = \{z : |z| < 1\}$ onto $\mathscr{D}$. The poincaré metric of $\mathscr{D}$ is defined by

(1) $$\lambda_{\mathscr{D}}(f(z))\,|f'(z)| = \lambda_{\mathscr{B}}(z) = 1/(1 - |z|^2), \quad z \in \mathscr{B}.$$

This definition is independent of the choice of conformal mapping and because of this convenient choices available. Namely, let $w \in \mathscr{D}$ and choose the conformal mapping so that $f(0) = w$. Then

(2) $$\lambda_{\mathscr{D}}(w) = 1/|f'(0)|.$$

If $f(z)$ is a conformal mapping of a domain $\mathscr{G}$ onto $\mathscr{D}$ then, from (1) and (2), we have

(3) $$\lambda_{\mathscr{D}}(f(z))\,|f'(z)| = \lambda_{\mathscr{G}}(z), \quad z \in \mathscr{G}.$$

Given $z \in \mathscr{D}$, let $d(z, \partial\mathscr{D})$ denote the distance from z to $\partial\mathscr{D}$, it is well-known that

(4) $$1/4 \leqq d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) \leqq 1, \quad z \in \mathscr{D}.$$

Osgood proved in [1] the following

Theorem A. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected and if f is analytic and univalent in $\mathscr{D}$ then*

(5) $$|f''(z)/f'(z)| \leqq 8\lambda_{\mathscr{D}}(z)$$

for all $z \in \mathscr{D}$. The inequality is sharp.

Theorem B. *If $\mathscr{D}$ is a proper subdomain of $\mathscr{C}$ and if f is analytic and univalent in $\mathscr{D}$ then*

(6) $$|f''(z)/f'(z)| \leqq 4/d(z, \partial\mathscr{D}), \quad z \in \mathscr{D}.$$

The inequality is sharp in the sense that the equality holds for $\mathscr{D} = \mathscr{B}$ and $f(z) = z/(1-z)^2$.

Our Theorem 1 generalizes the above Theorems A and B, which reveals the relationship of (5) and (6).

Theorem 1. *If $\mathscr{D} \subset \mathscr{C}$ is simply connected domain with at least two boundary points and $f(z)$ is analytic and univalent in $\mathscr{D}$ then*

(7) $$\left|\frac{f''(z)}{f'(z)}\right| \leqq \frac{4}{d(z, \partial\mathscr{D})}\left(4\sqrt{d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z)} - 2d(z, \partial\mathscr{D})\lambda_{\mathscr{D}}(z) - 1\right)$$

for all $z \in \mathscr{D}$, and the inequality is sharp.

For a differentiable function u we shall use the familiar operator

$$u_z = (u_x - iu_y)/2, \quad z = x + iy.$$

If $w = f(z)$ is a conformal mapping of a domain $\mathscr{G}$ onto $\mathscr{D}$ then from (3)

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