50. *Quadratic Irrationals, Ambiguous Classes and Symmetry in Real Quadratic Fields*

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Abstract: It is the purpose of this paper to set down the relationship between symmetry in the continued fraction expansion of a quadratic irrational, and the number of ambiguous ideals in an associated class of the class group of a real quadratic field. We also clear up some misconceptions in the literature pertaining to ambiguous classes.

In what follows we will establish the equivalence between real quadratic irrationals (with what we call pure symmetric period), and ambiguous classes having at most one ambiguous ideal in the class group of a real quadratic field. Although this should be well known, it is not set down anywhere in the literature. Moreover, what *is* set down is often misleading or simply wrong. We will point out some of these inaccuracies and set them straight.

First we need some background and notation.

Let D be a positive square-free integer and set

$$\omega = (\sigma - 1 + \sqrt{D})/\sigma$$

where $\sigma = 2$ if $D \equiv 1 (mod\ 4)$ and $\sigma = 1$ otherwise. The *discriminant* Δ of the real quadratic field $K = Q(\sqrt{D})$ is given by $\Delta = (2/\sigma)^2 D$. If $[\alpha, \beta]$ denotes the module $\{\alpha x + \beta y : x, y \in \boldsymbol{Z}\}$ then the *maximal order* (or ring of integers) O_Δ of K is $[1, \omega]$. The *norm* $N(\alpha)$ of $\alpha \in K$ is equal to $\alpha\alpha'$ where α' is the *algebraic conjugate* of α. The class group of K is denoted by C_Δ.

An ideal of O_Δ can be written as $I = [a, b + c\omega]$ where $a, b, c \in \boldsymbol{Z}$ with $a, c > 0$, $c \mid b$, $c \mid a$, and $ac \mid N(b + c\omega)$. Conversely, if $a, b, c \in \boldsymbol{Z}$ with $c \mid b$, $c \mid a$ and $ac \mid N(b + c\omega)$ then $[a, b + c\omega]$ is an ideal of O_Δ. In an ideal $I = [a, b + c\omega]$ with $a, c > 0$ the *norm of the ideal* I, $N(I)$ is given by $N(I) = ac > 0$. If $c = 1$ then I is said to be a *primitive* ideal. The *conjugate ideal* of $I = [a, b + \omega]$ is $I' = [a, b + \omega']$. An ideal I is called *reduced* if it is primitive and does not contain any non-zero element α such that both $|\alpha| < N(I)$, and $|\alpha'| < N(I)$. The class of an ideal I in O_Δ is denoted by $\{I\}$. For further details on the above, the reader is referred to [7].

At this juncture, we introduce continued fractions into the discussion. Given a *quadratic irrational* $\gamma \in K$ we may write $\gamma = (P + \sqrt{D})/Q$ where P, $Q \in \boldsymbol{Z}$ with $Q \neq 0$, and Q divides $N(P + \sqrt{D})$. Furthermore,

$$\gamma = \langle q_0, q_1, \ldots, q_i, \gamma_{i+1} \rangle$$

denotes the *continued fraction expansion* of γ where

$$\gamma_{i+1} = (P_{i+1} + \sqrt{D})/Q_{i+1}$$