41. Orders in Quadratic Fields. III

By R. A. MOLLIN

Mathematical Department, The University of Calgary, Canada

(Communicated by Shokichi IYANAGA, M. J. A., June 7, 1994)

Abstract: We disprove a conjecture posed in [3] concerning a criterion for the class group of complex quadratic orders to be generated by given ideal classes. Secondly we prove a necessary and sufficient condition for the class group C_Δ (for $\Delta < 0$) to be generated by ambiguous ideals in terms of the factorization of the Rabinowitsch polynomial. This shows that the well-known Rabinowitsch result [5] linking $h_\Delta = 1$ to the primality of Frobenius-Rabinowitsch polynomial for $\Delta < 0$ is not just a curiosity but rather underlies a deeper phenomenon.

The results herein continue the work of [3]-[4] to which we refer the reader for background and notation. The conjecture in [3, p. 48] says that the converse of [3, Theorem 2.1, p. 46] holds. We now show that this is, in fact, false. First, we need a useful technical result.

Lemma 1. *Let Δ be a discriminant with conductor f such that $(f, F_{\Delta,1}(x)) = 1$ for all integers $x \geq 0$. If a prime p divides Δ_0 then p^2 does not divide $F_{\Delta,1}(x)$ for any non-negative integers x.*

Proof. Since

$$4F_{\Delta,1}(x) = (2x + \sigma - 1)^2 - \Delta,$$

then $F_{\Delta,1}(x) \equiv 0 (mod\, p^2)$ implies that p^2 divides Δ ; whence, $p = 2$ and $\Delta \equiv 0 (mod\ 4)$. However, in this case, $F_{\Delta,1}(x) = x^2 - \Delta/4 = x^2 - f^2 D_0$ where $D_0 \equiv 2$ or $3\ (mod\ 4)$, a contradiction to $F_{\Delta,1}(x) \equiv 0 (mod\ 4)$.

The following example shows that the conjecture is false for $\Delta > 0$.

Example 1. Let $\Delta = \Delta_0 = 2^2 \cdot 5 \cdot 11 = 220$ where $D_0 = 55 \equiv 3 (mod$ $4)$, then $C_\Delta = \langle Q_2 \rangle$ where Q_2 is the unique ideal over 2, and $\lfloor M_\Delta \rfloor = 7$ ($\lfloor y \rfloor$ denotes the greatest integer less than or equal to y.) If the conjecture were true, then there would exist an integer $x \geqq 0$ and a $q \mid 2$ such that $\mid F_{\Delta,q}(x) \mid = 5r$ where r is not divisible by any unramified primes. By Lemma 1, $5r$ must be square-free; whence, $r \mid 22$. We further note that $F_{\Delta,1}(x) = x^2 - 55$ and $F_{\Delta,2}(x) = 2x^2 + 2x - 27$. To examine the possibilities we consider the following chart where we exhibit all $F_{\Delta,q}(x)$ such that $x \geq 0$, $q \mid 2, 5 \mid F_{\Delta,q}(x)$ and $\mid F_{\Delta,q}(x) \mid \leq 110$.

x	0	10
$\mid F_{\Delta,1}(x) \mid$	$5 \cdot 11$	$5 \cdot 9$

x	2	7
$\mid F_{\Delta,2}(x) \mid$	$5 \cdot 3$	$5 \cdot 17$

We observe that only $\mid F_{\Delta,1}(x) \mid = 5 \cdot 11$ fits the criterion. However, the