38. *Index for Factors Generated by Direct Sums of II_1 Factors*

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In this paper, we give an index formula for II_1 factors generated by increasing sequences of infinite dimensional algebras and some examples of such factors. The theory in case of finite dimensional algebras was constructed by H. Wenzl.

§1. Preliminaries. Let $M = \oplus_{j=1}^{m} M_j$ be a finite direct sum of II_1 factors and q_j the minimal central projection corresponding to M_j. Since the normalized normal trace on a II_1 factor is unique, a trace on M (denoted by tr) is decided by a numerical vector $\vec{s} = (tr(q_i))_{i=1,\dots,m}$, called the trace vector of M. Let $N = \oplus_{i=1}^{n} N_i \subset M$ be an another finite direct sum of II_1 factors and p_i the corresponding minimal central projection. We assume that the trace on N is the restriction of the trace on M, and denote by $\vec{t}$ the trace vector of N.

We define two matrices relating the inclusion relation $N \subset M$, the index matrix and the trace matrix. The index matrix $\Lambda_N^M = (\lambda_{ij})$ is given by

$$\lambda_{ij} = \begin{cases} [M_{p_i q_j} : N_{p_i q_j}]^{1/2} & p_i q_j \neq 0, \\ 0 & p_i q_j = 0, \end{cases}$$

and the trace matrix $T_N^M = (t_{ij})$ by $t_{ij} = tr_{M_j}(p_i q_j)$, where tr_{M_j} is the unique normalized normal trace on M_j.

We suppose that N is of finite index in M, i.e., there is a faithful representation π of M on a Hilbert space such that $\pi(N)'$ is finite. Then the algebra $\langle M, e_N \rangle$ obtained by basic construction for $N \subset M$ is a finite direct sum of II_1 factors and the corresponding minimal central projections are $Jq_1J, \dots, Jq_mJ$, where J is the canonical conjugation on $L^2(M, tr)$. We know the following in [1].

(1.1) The equality $\vec{t} = T_N^M \vec{s}$ holds.

(1.2) $$\Lambda_M^{\langle M, e_N \rangle} = (\Lambda_N^M)^t$$

(1.3) $$T_M^{\langle M, e_N \rangle} = \tilde{T}_N^M F_N^M,$$

where $(\tilde{T}_N^M)_{ji} = \begin{cases} \dfrac{\lambda_{ij}^2}{t_{ij}} & p_i q_j \neq 0 \\ 0 & p_i q_j = 0, \end{cases}$, $F_N^M = diag(\varphi_1, \cdots, \varphi_n)$, $\varphi_i = (\sum_j (\tilde{T}_N^M)_{ji})^{-1}$.

(1.4) For any trace Tr on $\langle M, e_N \rangle$, $Tr(e_N p_i) = \varphi_i Tr(Jp_iJ)$.

The index $[M : N]$ is defined as $[M : N] = r(\tilde{T}_N^M T_N^M)$, where $r(T)$ is the spectral radius of T.

Now let $M_0 \subset M_1$ be a pair of II_1 factors with finite index and trivial relative commutant. By the basic construction, we obtain a tower of II_1 fac-