5. Graded Algebras of Vector Bundle Maps over an Elliptic Curve

By Daisuke TAMBARA

Department of Mathematics, Hirosaki University

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We study here a kind of homogeneous coordinate rings of matrix algebras over an elliptic curve. Let X be an elliptic curve over an algebraically closed field k with $\mathrm{char}(k) \neq 2$. Choose a point $P \in X$ and let $\mathscr{L} = \mathscr{L}(P)$ be the invertible $\mathscr{O}_X$-module associated to the divisor P. For a positive integer n let $\mathscr{E}_n$ be an indecomposable locally free $\mathscr{O}_X$-module of rank n which is a successive extension of $\mathscr{O}_X$. Such a module exists uniquely up to isomorphism ([2]). We form the $\mathscr{O}_X$-algebra $\mathscr{E}nd(\mathscr{E}_n)$, the sheaf of local endomorphisms of $\mathscr{E}_n$, and then form a graded k-algebra

$$\Lambda(n) = \bigoplus_{i\geq 0} \Gamma(X, \mathscr{E}nd(\mathscr{E}_n) \otimes \mathscr{L}^{\otimes i}) = \bigoplus_{i\geq 0} \mathrm{Hom}(\mathscr{E}_n, \mathscr{E}_n \otimes \mathscr{L}^{\otimes i}).$$

In this paper we give an explicit description of the algebra $\Lambda(n)$. Details and proofs will appear elsewhere.

1. Realization of $\Lambda(n)$ as a matrix algebra. Put $S = \bigoplus_{i\geq 0} \Gamma(X, \mathscr{L}^{\otimes i})$. This is a commutative graded k-algebra. For an $\mathscr{O}_X$-module $\mathscr{F}$ we put $\Gamma_*(\mathscr{F}) = \bigoplus_{i\in \mathbf{Z}} \Gamma(X, \mathscr{F} \otimes \mathscr{L}^{\otimes i})$, which is a graded S-module. Also $\Lambda(n)$ is an S-algebra. Since $\mathscr{L}$ is ample, we have $\Lambda(n) \cong \mathrm{End}_S(\Gamma_*(\mathscr{E}_n))$ as S-algebras (cf. [1]).

The algebra S is generated by suitable homogeneous elements t, x, y of degree 1, 2, 3, respectively, with relation $y^2 = x(x - t^2)(x - \lambda t^2)$ for some $\lambda \in k - \{0, 1\}$ ([3, p. 336]). We fix t, x, y, λ throughout. Put $v = x - (\lambda + 1)t^2$, $u = (x - t^2)(x - \lambda t^2)$.

Let $R = k[t, x]$, a polynomial subalgebra of S. Then $S = R \oplus Ry$. Define a graded S-module M as follows. M is a free graded R-module with basis $\alpha, \beta_i, \gamma_i$ for $i > 0$ with $\deg \alpha = 0$, $\deg \beta_i = 1$, $\deg \gamma_i = 2$. The action of y on M is given by

$$y\alpha = x\beta_1 + t\gamma_1$$
$$y\beta_i = -\lambda t^3 O_i \beta_{i-1} - tx\beta_{i+1} + v\gamma_{i-1} - t^2\gamma_{i+1}$$
$$y\gamma_i = x^2\beta_{i+1} + \lambda t^3 E_i \gamma_{i-1} + tx\gamma_{i+1}$$

where $\beta_0 = -t\alpha$, $\gamma_0 = x\alpha$ and $O_i = 1$ for an odd i, $O_i = 0$ for an even i, $E_i = 1 - O_i$. For $n \geq 1$ define a graded S-submodule $M(n)$ of M to be the free R-submodule generated by $\alpha, \beta_i, \gamma_i$ for $1 \leq i \leq n-1$ and $x\beta_n + t\gamma_n$.

Proposition 1. *$\Gamma_*(\mathscr{E}_n) \cong M(n)$ as graded S-modules.*

So we can identify $\Lambda(n) = \mathrm{End}_S(M(n))$.

Though the S-module M is not free, the $S\left[\frac{1}{y}\right]$-module $M\left[\frac{1}{y}\right] = S\left[\frac{1}{y}\right] \otimes_S M$ is free with basis α_i, $i \geq 0$, given by $\alpha_i = \frac{1}{x}\gamma_i$ if i is odd, $\alpha_i =$