26. *On a Conjecture of Shanks*

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§1. Introduction. The purpose of the present article is to give some refinements of the previous works [2]-[6] concerning Shanks' conjecture.

Let $\rho = \beta + i\gamma$ run over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. In explaining theoretically a strange tendency which appears when one draws the graph of $\zeta\left(\frac{1}{2} + it\right)$ for $t \geq 0$ in the complex plain, Shanks [8] has given the following conjecture.

Conjecture. $\zeta'\left(\frac{1}{2} + i\gamma\right)$ *is positive real in the mean.*

Concerning this, we can show the following theorems. We suppose always that $T > T_o$ and C denotes some positive constant. Let R. H. be the abbreviation of the Riemann Hypothesis. Let C_o and C_1 be the Laurent coefficients in

$$\zeta(s) = \frac{1}{s-1} + C_o + C_1(s-1) + \dots .$$

Theorem 1.
$$\sum_{0<\gamma\leq T} \zeta'(\rho) = \frac{1}{4\pi} T \log^2 \frac{T}{2\pi} + (C_o - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (C_1 - C_o)\frac{T}{2\pi} + O(T\exp(-C\sqrt{\log T})).$$

Theorem 2 (Under R. H.).
$$\sum_{0<\gamma\leq T} \zeta'\left(\frac{1}{2} + i\gamma\right) = \frac{1}{4\pi} T \log^2 \frac{T}{2\pi} + (C_o - 1)\frac{T}{2\pi}\log\frac{T}{2\pi} + (C_1 - C_o)\frac{T}{2\pi} + O(T^{\frac{1}{2}}\log^{\frac{7}{2}} T).$$

These imply that $\zeta'(\rho)$ is positive real in the mean and also improve upon both our previous results [2][4][6] and also Conrey-Gohsh-Gonek [1]. Theorem 1 is announced in [6].

On the other hand, the following two theorems may provide us an explanation of the strange tendency mentioned above.

Theorem 3. *For* $0 \neq \Delta = 2\pi\alpha/\log(T/2\pi) \ll 1$, *we have*

$$\sum_{0<\gamma\leq T} \zeta(\rho + i\Delta) = \pi\alpha\left(\frac{1 - \frac{\sin 2\pi\alpha}{2\pi\alpha}}{\pi\alpha} + i\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+ \frac{T}{2\pi}\left(-1 + \left(\frac{T}{2\pi}\right)^{-i\Delta}\frac{1}{i\Delta}\left(\frac{1}{1-i\Delta} - 1\right) - \left(\frac{T}{2\pi}\right)^{-i\Delta}\frac{1}{1-i\Delta}\left(\zeta(1-i\Delta) + \frac{1}{i\Delta}\right)\right.$$
$$\left. + \left(\frac{\zeta'}{\zeta}(1+i\Delta) + \frac{1}{i\Delta}\right)\right) + O(T\exp(-C\sqrt{\log T})).$$