25. *Triangles and Elliptic Curves*[*)]

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In this paper, we shall obtain a family of infinitely many elliptic curves defined over an algebraic number field k so that every curve in it has positive Mordell-Weil rank with respect to k. The construction of curves is very easy: we have only to replace *right* triangles in the antique congruent number problem by *arbitrary* triangles.

§1. Arbitrary field. Let k be a field of characteristic $\neq 2$ and let $\bar{k}$ be an algebraic closure of k, fixed once for all. For three elements a, b, c in $\bar{k}$, we shall put

$$P = \frac{1}{2}(a^2 + b^2 - c^2), \tag{1.1}$$

$$\begin{aligned} Q &= \frac{1}{16}(a+b+c)(a+b-c)(a-b+c)(a-b-c) \\ &= \frac{1}{16}(a^4 + b^4 + c^4 - 2a^2b^2 - 2b^2c^2 - 2c^2a^2). \end{aligned} \tag{1.2}$$

One verifies easily that

$$P^2 - 4Q = a^2b^2. \tag{1.3}$$

Now consider the plane cubic:

$$y^2 = x^3 + Px^2 + Qx = x\left(x + \frac{P+ab}{2}\right)\left(x + \frac{P-ab}{2}\right). \tag{1.4}$$

From (1.3), (1.4), one finds that the cubic is non-singular if and only if

$$abQ \neq 0. \tag{1.5}$$

We shall call E the elliptic curve given by (1.4) with the condition (1.5). Referring to standard definitions on Weierstrass equations ([1] p. 46), we find the values of the discriminant and the j-invariant of E in terms of a, b, c, P, Q:

(1.6) $\Delta = (4abQ)^2 = 16D$, D being the discriminant of $x^3 + Px^2 + Qx$,

(1.7) $j = 2^8(P^2 - 3Q)^3/(abQ)^2 = 2^8(Q + a^2b^2)^3/(abQ)^2$.

(1.8) **Remark.** Although not neccesary in this paper, we mention here a basic fact. A simple calculation shows that if (a, b, c) and (a', b', c') are triples in $\bar{k}$ with (1.5) such that $a' = ra$, $b' = rb$, $c' = rc$ with $r \in \bar{k}^{\times}$, then they have the same j-invariant. Consequently, our construction $(a, b, c) \mapsto E$ induces a map:

(1.9) $P^2(\bar{k}) - H \rightarrow \bar{k}$ (moduli space of elliptic curves over $\bar{k}$),

where H is the union of six lines $a = 0$, $b = 0$, $a + b + c = 0$, $a + b - c = 0$, $a - b + c = 0$ and $a - b - c = 0$.

(1.10) **Lemma.** *Let E be the elliptic curve defined by $a, b, c \in \bar{k}$ with* (1.5).

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*) Dedicated to Professor S. Iyanaga on his 88th birthday.