19. A Continuation Principle for the 3-D Euler Equations for Incompressible Fluids in a Bounded Domain

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1. In this paper we study the Euler equations for ideal incompressible fluids in a bounded domain Ω in $\boldsymbol{R}^3$:

$$(1)\qquad u_t + u\cdot\nabla u + \nabla p = 0, \quad \nabla\cdot u = 0 \text{ for } t \geq 0,\ x \in \Omega,$$

$$(2)\qquad u\cdot n = 0 \quad \text{for} \quad t \geq 0,\ x \in \Gamma.$$

Here the boundary Γ of Ω is assumed to be of class C^∞; t and x are time and space variables; $u = u(t, x) = (u_1, u_2, u_3)$ is the velocity and $p = p(t, x)$ is the pressure; $n = n(x) = (n_1, n_2, n_3)$ is the unit outward normal at $x \in \Gamma$; we write $u_t = \partial u/\partial t$, $\partial_i = \partial/\partial x^i$ for $i = 1, 2, 3$, $\nabla = (\partial_1, \partial_2, \partial_3)$ and $u\cdot\nabla = \sum_{i=1}^3 u_i\partial_i$.

Let $s \geq 0$ be an integer. We denote by $H^s(\Omega\,;\boldsymbol{R}^3)$ the usual Sobolev space of order s on Ω taking values in $\boldsymbol{R}^3$. The norm is defined by $\|u\|_s^2 = \sum_{|\alpha|\leq s} |\partial^\alpha u|_{L^2(\Omega)}^2$, where $\partial^\alpha = \partial^{|\alpha|}/\partial_1^{\alpha_1}\partial_2^{\alpha_2}\partial_3^{\alpha_3}$ with $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. For $0 < T < \infty$, we put

$$X_s(T) = C^0([0, T]\,; H^s(\Omega\,;\boldsymbol{R}^3)) \cap C^1([0, T]\,; H^{s-1}(\Omega\,;\boldsymbol{R}^3)).$$

Now we state our main

Theorem. *Let $s > 2$ be an integer. Suppose that u is a solution of* (1), (2) *belonging to $X_s(T')$ for any $T' < T < \infty$ such that $\|u(t)\|_s \uparrow \infty$ as $t \uparrow T$. Then*

$$(3)\qquad \int_0^t |\operatorname{rot} u(\tau)|_{L^\infty(\Omega)} d\tau \uparrow \infty \text{ as } t \uparrow T.$$

This theorem is an immediate consequence of the local in time existence theorem for the initial boundary value problem (1), (2) with the initial data $u^0 \in H^s(\Omega\,;\boldsymbol{R}^3)$ satisfying $\nabla\cdot u^0 = 0$ in Ω, $u^0\cdot n = 0$ on Γ (see [3,6]), and the following new estimate for a smooth solution u of (1), (2) such that $u \in X_s(T)$ with $s > 2$: There exists a nondecreasing continuous function $F(t, x, y) \geq 0$ for $t \geq 0$, $x \geq 0$, $y \geq 0$, satisfying the estimate

$$(4)\qquad \|u(t)\|_s \leq F(t, \|u(0)\|_s, \int_0^t |\operatorname{rot} u(\tau)|_{L^\infty(\Omega)} d\tau) \quad \text{for } t \in [0, T].$$

In the sequel, C is a constant which might change line by line and $u(t, x)$ is always a smooth solution of (1), (2) in the sense mentioned above.

Such a link that exists between the accumulation of the vorticity and the possible breakdown of smooth solutions for the Euler equations was shown by Beale-Kato-Majda [2] for the motion of fluids in the entire space $\boldsymbol{R}^3$.

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