15. On Non-starlikeness of Teichmüller Spaces

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(Communicated by Kiyosi ITÔ, M. J. A., March 12, 1993)

Abstract: Recently, Krushkal showed that the universal Teichmüller space is not starlike. In this note, we shall extend his result to Teichmüller spaces of Fuchsian groups whose quotient surfaces have an arbitrarily large disc. Especially, Teichmüller spaces of Fuchsian groups of the second kind are not starlike.

Let H and H^* be the upper half plane and the lower half plane, respectively. The hyperbolic distance between z and w in H is denoted by $d(z, w)$.

Let Γ be a Fuchsian group acting on H, i.e., a discrete subgroup of $PSL(2,\boldsymbol{R})$. The trivial group is denoted by 1. We say that a set X is precisely invariant under the trivial group 1 in Γ if $\gamma(X) \cap X = \phi$ for all $\gamma \in \Gamma - \{id.\}$.

Let $Q(\Gamma)$ be the complex Banach space of bounded holomorphic quadratic differentials for Γ on H^*. Namely, $Q(\Gamma)$ is the set of holomorphic functions φ on H^*, with norm

$$\|\varphi\| = \sup_{z\in H^*} |4y^2\varphi(z)| < \infty \ (z = x + iy),$$

and satisfying the functional equation

$$\varphi(\gamma(z))\gamma'(z)^2 = \varphi(z),\ \gamma \in \Gamma.$$

For each $\varphi \in Q(\Gamma)$, there exists a locally schicht holomorphic function W_φ on H^* such that the Schwarzian derivative $\{W_\varphi, z\}$ is equal to $\varphi(z)$. The Teichmüller space $T(\Gamma)$ of Γ is the set of $\varphi \in Q(\Gamma)$ such that W_φ admits a quasiconformal extention to the Riemann sphere $\hat{C}$. Moreover, we denote by $S(\Gamma)$ the set of those $\varphi \in Q(\Gamma)$ for which W_φ are schlicht.

A subset M of $Q(\Gamma)$ is said to be starlike with respect to $\varphi \in M$ if for every $\psi \in M$,

$$\{(1-t)\varphi + t\psi\ ;\ o \le t \le 1\} \subset M.$$

Our main theorem is:

Theorem 1. *Let Γ be a Fuchsian group acting on H. Suppose that H/Γ has an arbitrarily large disc, i.e., for any positive number r, there exists a hyperbolic disc of radius r which is precisely invariant under the trivial group in Γ. Then $T(\Gamma)$ is not starlike with respect to any point of it.*

Corollary. *Teichmüller spaces of Fuchsian groups of the second kind are not starlike with respect to any of their points.*

We need the following proposition which was essentially proved in [5].

Proposition. *Let $\Gamma_n (n \in N)$ be Fuchsian groups acting on H. Suppose that there is a sequence $\{r_n\}_{n=1}^\infty$ of positive numbers such that $\Delta(r_n) = \{z \in H\ ;\ d(z, i) < r_n\}$ is precisely invariant under the trivial group in Γ_n for each n and*