# 92. *On the Local Regularity of Solutions to the Simultaneous Relations Characterizing the Supporting Functions of Convex Curves of Constant Angle*

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**Abstract**: We shall define a curve of constant angle $\alpha$, $0 < \alpha < \pi$ in the plane $\boldsymbol{R}^2$. This curve is a closed convex curve parametrized by $\theta \in \boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$ and characterized by a $C^1$ function $p(\theta)$ called the *supporting function*. We shall show that $\ddot{p}(\theta)$, the second derivative of $p(\theta)$ in the sense of distributions of L. Schwartz, belongs to $L^\infty$. This result is the best possible one if the angle $\alpha$ is general.

**Key words**: local regularity; supporting function.

**1. Characteristic function $\chi_\alpha$ and modified characteristic function $\tilde{\chi}_\alpha$.** Let $\alpha$ be a given angle $0 < \alpha < \pi$. Put $\hat{\alpha} = \pi - \alpha$. We use the notations

(1.1) $\quad c_1(\alpha) = \sin\alpha,\ c_2(\alpha) = \cos\alpha,\ \tilde{c}_1(\alpha) = \sin\alpha/2,\ \tilde{c}_2(\alpha) = \cos\alpha/2$

and we omit the variable as far as there is no confusion. Let $\Omega_\alpha = \min\{\tilde{c}_1, \tilde{c}_2\}$. The open intervals $I_\alpha$ and $J_\alpha$ are defined as follows:

(1.2) $$I_\alpha = (-\Omega_\alpha, \Omega_\alpha),$$

(1.3) $$J_\alpha = \begin{cases} (0, c_1) & \text{for } 0 < \alpha \leq \pi/2 \\ (-c_2, 1) & \text{for } \pi/2 \leq \alpha < \pi. \end{cases}$$

The *characteristic function* $\chi_\alpha$ and the *modified characteristic function* $\tilde{\chi}_\alpha$ are defined by the formulas

(1.4) $$\chi_\alpha(t) = c_1(1-t^2)^{1/2} - c_2 t,\ t \in J_\alpha;$$

(1.5) $$\tilde{\chi}_\alpha(s) = \tilde{c}_1(1-s^2)^{1/2} - \tilde{c}_2 s,\ s \in I_\alpha \quad \text{or} \quad s \in J_\alpha.$$

We state some properties of these functions without proofs.

**Proposition 1.1.** *$\chi_\alpha$ maps $J_\alpha$ onto $J_\alpha$ and is strictly monotone decreasing. $\chi_\alpha$ has the only one fixed point $\tilde{c}_1$. Its inverse mapping $\overset{-1}{\chi_\alpha}$ coincides with $\chi_\alpha$. $\tilde{\chi}_\alpha$ maps $J_\alpha$ onto $I_\alpha$ and is strictly monotone decreasing. $\tilde{\chi}_\alpha$ maps $\tilde{c}_1$ to 0. Its inverse mapping $\overset{-1}{\tilde{\chi}_\alpha}$ has the same expression as $\tilde{\chi}_\alpha$.*

$\tilde{\chi}_\alpha$ has the linearization effect on $\chi_\alpha$ as follows:

**Proposition 1.2.** *If $w$ belongs to $I_\alpha$, $p$ belongs to $J_\alpha$, and $w = \tilde{\chi}_\alpha(p)$, then $\tilde{\chi}_\alpha(\chi_\alpha(p)) = -w$.*

**2. Curves of constant angle $\alpha$.** Let $C$ be the circle of radius $r$ with the center at the origin of the plane $\boldsymbol{R}^2$, and call it the *director circle*. (This terminology comes from the classical example of ellipses, that is, $\alpha = \pi/2$.) Hereafter we assume $r = 1$, without loss of generality. Let $A$ be a figure contained in $C$. A figure simply means here a subset of $\boldsymbol{R}^2$. For a point $P$ on $C$, we put

$$C(P; A) = \{\text{ray; starting from } P, \text{ passing through a point of } A\},$$