# 85. *On the Neumann Problems for Certain Degenerate Elliptic Operators**),**)

By Toshio HORIUCHI

Department of Mathematics, Ibaraki University

(Communicated by Kiyosi ITÔ, M. J. A., Nov. 12, 1993)

**§1. Introduction.** Let $\Omega$ be a bounded domain in $\boldsymbol{R}^n (n \geq 2)$ with $\partial\Omega \in C^\infty$. Let $\varphi$ be a given nonnegative smooth function on $\bar{\Omega}$ and equivalent to a distance to the boundary:

$$\Omega = \{\varphi(x) > 0\},\ \partial\Omega = \{\varphi(x) = 0\},\ d\varphi \neq 0 \quad \text{on } \partial\Omega.$$

Let $\alpha(x)$ be a $C^2$-function on $\bar{\Omega}$. In the Dirichlet problem (D-P), we impose on $\alpha$ the following condition (D).

$$\text{(D)} \qquad \frac{d\alpha}{dn} = 0 \quad \text{on } \partial\Omega,\ \boldsymbol{n}: \text{unit normal.}$$

The purpose of this note is to study the homogeneous Dirichlet and Neumann boundary problems defined by

$$\text{(D-P)} \qquad \begin{cases} -\operatorname{div}(\varphi^\alpha \nabla u) + c(x)\varphi^{\alpha-1}u = f & \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \quad \text{We assume (D).} \end{cases}$$

$$\text{(N-P)} \qquad \begin{cases} -\operatorname{div}(\varphi^\alpha \nabla u) + c(x)\varphi^\alpha u = f & \text{in } \Omega \\ \dfrac{du}{dn} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

As for (D-P), we have

**Proposition 1.** *Suppose that $\alpha$ satisfies* (D) *and $\alpha < 1$ on $\partial\Omega$. Then, there exists a positive number $M$ such that if $\inf_{x\in\Omega} c(x) \geq M$, then for every $f \in C^\mu(\bar{\Omega})$, there exists one and only one solution $u$ to* (D-P) *which is written as $u = \varphi^{1-\alpha}v$ with a function $v$ belonging to $C^{1+\mu}(\bar{\Omega})$ such that $\varphi v \in C^{2+\mu}(\bar{\Omega})$.*

From the theory of ordinary differential equations, we see that (D-P) has no solution vanishing at $x_n = 0$ if $\alpha \geq 1$. So, the condition $\alpha < 1$ is necessary for us. If we set $u = \varphi^{1-\alpha}v$ in (D-P), this proposition follows as a corollary to the theorem due to C. Goulaouic-N. Shimakura [1] in which the equation $\varphi\Delta v + z\partial_n v + f = 0 (f \in C^\mu(\bar{\Omega}), \Re z > 0)$ were studied. Note that the restriction (D) can be relaxed as follows:

$$\alpha \in C^2(\Omega),\quad |(\nabla\varphi,\ \nabla\alpha)|,\ \varphi|\Delta\alpha| = O(\varphi^\delta), \text{ for some } \delta > \mu.$$

**§2. Preliminaries to the Neumann problem.** In the Neumann problem, we have to deal with unbounded solutions as we shall show in §4. So, we are obliged to modify the classical Schauder spaces to admit unbounded solutions.

**Definition.** Let $0 < \mu < 1$, $\beta \neq 0$, and let $f \in C(\boldsymbol{R}^n_+)$ be compactly

---

*) Dedicated to Professor Morisuke Hasumi on his sixtieth birthday.

**) This research was partially supported by Grant-in-Aid for Scientific Research (No. 04854005), Ministry of Education, Science and Culture.