83. *Group Rings and the Norm Groups*

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1. Introduction and preliminary lemmas. Let n be a natural number >1 and G be a cyclic group of order n generated by σ. We consider in this note the cyclic extension L/F of fields with the Galois group G. Let $a \in L^{\times}$. The well-known Hilbert theorem 90 asserts that $a^{1+\sigma+\cdots+\sigma^{n-1}} = 1$ if and only if there exists $b \in L^{\times}$ such that $a = b^{1-\sigma}$. Now let t be an indeterminate and set $D_n = \{f(t) \in \boldsymbol{Z}[t] \mid f(t) \text{ divides } t^n - 1\}$. For $f(t) \in D_n$, we shall denote $f^{\perp}(t) = (t^n - 1)/f(t)$. Obviously one sees $f^{\perp}(t) \in D_n$ and $(f^{\perp})^{\perp}(t) = f(t)$. We define now:

(1. 1) $f(t) \in D_n$ is called of *H-type* if the following holds:
For any cyclic extension L/F and any $a \in L^{\times}$, $a^{f(\sigma)} = 1$ if and only if there exists $b \in L^{\times}$ such that $a = b^{f^{\perp}(\sigma)}$.
If there is no fear of confusion, we shall abbreviate $f(t)$ or $f(\sigma)$ to f. It is obvious that $a = b^{f^{\perp}}$ implies $a^{f} = 1$, so that the above definition can be simplified as follows:

(1.2) f is of *H-type*, if $a^{f(\sigma)} = 1$ implies the existence of b with $a = b^{f^{\perp}(\sigma)}$.
$f = t^n - 1$ is trivially of *H-type*, and Hilbert theorem 90 says that $f = 1 + t + \cdots + t^{n-1}$ is of *H-type*. W. Hürlimann [2] has proved an interesting result ("Cyclotomic Hilbert theorem 90") saying that the n-th cyclotomic polynomial $\Phi_n(t)$ is also of *H-type*.

The aim of this paper is to determine the set of all polynomials ($\in D_n$) of *H-type*, which will be denoted with H_n. The result of [2] will be stated as

Lemma 1. $\Phi_n \in H_n$.

We denote the greatest common divisor and the least common multiple of f, $g \in \boldsymbol{Z}[t]$ by (f, g) and $\{f, g\}$, respectively. If $f, g \in D_n$ we have clearly $(f, g), \{f, g\} \in D_n$.

Lemma 2. *If $f, g \in D_n$ are of H-type, then (f, g) and $\{f, g\}$ are of H-type.*

Proof. We denote $f_0 = (f, g)$ and $f = f_0 f_1$, $g = f_0 g_1$ and $t^n - 1 = f_0 f_1 g_1 h$. We shall show $f_0 = (f, g)$ is of *H-type*. For any $a \in L^{\times}$ such that $a^{f_0} = 1$, one sees $a^{f} = 1$. Since f is of *H-type*, there exists $b \in L^{\times}$ such that $a = b^{g_1 h}$. Then $a^{f_0} = (b^h)^g = 1$. Since, g is of *H-type*, there exists $c \in L^{\times}$ such that $b^h = c^{f_1 h}$. Hence $a = (b^h)^{g_1} = c^{f_1 g_1 h} = c^{f_0^{\perp}}$. In the same way as above, one sees that $\{f, g\}$ is also of *H-type*.

For the case $m \mid n$, we define an injection $\pi_{n/m}$ from D_m to D_n by putting $\pi_{n/m}(f(t)) = f(t^l)$, where $l = n/m$. We shall abbreviate $\pi_{n/m}(f(t))$ to