71. On the Intersection of Continuous Local Tents

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Let X be an n-dimensional Euclidean space with inner product $(-\mid -)$ and norm $|-|$. The open unit ball $\{x \in X\,;\,|x| < 1\}$ is denoted by B. For a convex set $M \subset X$ we denote by aff M the smallest affine subspace of X containing M. Then there exists a unique vector subspace lin M of X such that aff $M = x +$ lin M for any $x \in$ aff M. The interior of M with respect to the subspace aff M is called the *relative interior* of M and denoted by rint M.

Next, let $K_1, \ldots, K_s$ be convex cones with common vertex at x_0. The system $K_1, \ldots, K_s$ is said to be *separable* if one of these cones can be separated by a hyperplane from the intersection of the others.

Further, let $\Omega \subset X$ and $x_0 \in \Omega$. A convex cone $Q \subset X$ with vertex at x_0 is called a *tent* (*continuous tent, smooth tent*) at x_0 of Ω if there exists a mapping (continuous mapping, smooth mapping) $\psi : X \to X$ defined in a neighborhood of x_0 such that

1) $\psi(x) = x + o(x - x_0)$, where $\lim\limits_{\xi \to 0} \dfrac{o(\xi)}{|\xi|} = 0$ and $o(0) = 0$,

2) for some $\varepsilon > 0$, $\psi(Q \cap (x_0 + \varepsilon B)) \subset \Omega$.

More generally, a convex cone $K \subset X$ with vertex at x_0 is called a *local tent* (*continuous local tent, smooth local tent*) if, for any $\bar{x} \in K$, there exists a tent (continuous tent, smooth tent) $Q \subset K$ at x_0 of Ω such that $\bar{x} \in$ rint Q and aff $Q =$ aff K.

The notion of continuous and smooth tents (or local tents) is effectively used in deriving necessary conditions for optimality in various optimization problems. For obtaining these necessary conditions, fundamental roles are played by the following

Theorem A. *Let $\Omega_1, \ldots, \Omega_s$ be subsets of X having a point x_0 in common and $K_1, \ldots, K_s$ an inseparable system of continuous local tents at x_0 of $\Omega_1, \ldots, \Omega_s$, respectively. Assume that $K_j \neq$ off K_j for some j. Then there exists a point $x_1 \in \Omega := \cap_{j=1}^{s} \Omega_j$ other than x_0.*

By a subtle algebraic topological method, V. G. Boltjanskiĭ ([1]) gave a proof of Theorem A, printed in small type on a total of 14 pages.

Theorem A, however, is a simple consequence of the following

Theorem B (theorem on the intersection of continuous local tents). *Let $\Omega_1, \ldots, \Omega_s$ be subsets of X having a point x_0 in common and $K_1, \ldots, K_s$ an inseparable system of continuous local tents at x_0 of $\Omega_1, \ldots, \Omega_s$, respectively. Then $K := \cap_{j=1}^{s} K_j$ is a* (*not necessarily continuous*) *local tent at x_0 of $\Omega := \cap_{j=1}^{s} \Omega_j$.*

In fact, we may assume $x_0 = 0$. If $K_{j0} \neq$ lin K_{j0} and $0 \in$ rint K_{j0}, then