70. *Explicit Formulas and Asymptotic Expansions for Certain Mean Square of Hurwitz Zeta-functions*

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Let $\zeta(s, \alpha)$ be the Hurwitz zeta-function with a positive parameter α, and $\zeta_1(s, \alpha) = \zeta(s, \alpha) - \alpha^{-s}$. Recently, three proofs of the conjecture

$$\int_0^1 \left| \zeta_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha = \log(t/2\pi) + \gamma + O(t^{-\frac{1}{4}}), \tag{1}$$

where $t \geq 1$ and γ is Euler's constant, have appeared. Zhang's proof [10] is based on the functional equation of $\zeta(s, \alpha)$, and actually, he proved the following stronger result:

$$\int_0^1 \left| \zeta_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha = \log(t/2\pi) + \gamma - 2\mathrm{Re}\, \frac{\zeta\left(\frac{1}{2} + it\right)}{\frac{1}{2} + it} + O(t^{-1}), \tag{2}$$

where $\zeta(s)$ is the Riemann zeta-function. Another proof of (2) is given in Andersson [1], who obtained certain explicit formulas (Corollaries 1 and 2 below) which implies (2). His proof is based on Mikolás' idea [7] of using Parseval's identity. The third proof, sketched in the authors' article [6], is a variant of Atkinson's method, and the key lemma is the explicit formula [6, (3.1)]. The main idea of this proof is based on the works of Atkinson [2], Motohashi [8] and the authors [4].

By refining the argument of the third proof, we can prove several explicit formulas and asymptotic expansions, which we announce in this note. The proofs will appear elsewhere.

The first result is a further refinement of Andersson-Zhang's formula (2). Let $\Gamma(s)$ be the gamma-function, and $\psi(s) = (\Gamma'/\Gamma)(s)$. Then,

Theorem 1. *For any integer $K \geq 0$, we have the asymptotic expansion*

$$\int_0^1 \left| \zeta_1\left(\frac{1}{2} + it, \alpha\right) \right|^2 d\alpha$$
$$= \gamma - \log 2\pi + \mathrm{Re}\, \psi\left(\frac{1}{2} + it\right) - 2\,\mathrm{Re}\, \frac{\zeta\left(\frac{1}{2} + it\right) - 1}{\frac{1}{2} + it}$$
$$- 2\,\mathrm{Re} \sum_{k=1}^{K} \frac{(-1)^{k-1}(k-1)!}{\left(\frac{3}{2} - k + it\right)\left(\frac{5}{2} - k + it\right)\cdots\left(\frac{1}{2} + it\right)} \sum_{l=1}^{\infty} l^{-k}(l+1)^{-\frac{3}{2}+k-it}$$
$$+ O(t^{-K-1}).$$

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