68. On the Existence of Characters of the Schur Index 2 of the Simple Finite Steinberg Groups of Type $(^2E_6)$*)

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Let χ be a complex irreducible character of a finite group and k be a field of characteristic 0. Then we denote by $m_k(\chi)$ the Schur index of χ with respect to k.

It has been known that the simple group $\mathrm{PSU}(3, q^2)$ has an irreducible character χ with $m_{\boldsymbol{Q}}(\chi) = 2$ (R. Gow [4]). In [5], (7.6), G. Lusztig found that $\mathrm{PSU}(3, q^2)$ or $\mathrm{PSU}(6, q^2)$ has a rational-valued irreducible character χ such that $m_{\boldsymbol{Q}}(\chi) = m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ (q is a power of p) and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$. For $\mathrm{PSU}(3, q^2)$, this χ coincides with the one described above. In this note we shall show that the simple finite Steinberg group $^2E_6(q^2)$ has (at least) two rational-valued irreducible characters χ such that $m_{\boldsymbol{Q}}(\chi) = m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$. This will follow from Lusztig's classification theory of the unipotent representations of finite groups of Lie type (see [2], pp. 480–481).

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Let $\boldsymbol{F}_q$ be a finite field with q elements, of characteristic p. If X is an algebraic group defined over $\boldsymbol{F}_q$, then $X(q)$ denotes the group of $\boldsymbol{F}_q$-rational points of X. Then we have

Lemma. *Let M be a connected, reductive algebraic group, defined over $\boldsymbol{F}_q$, whose Coxeter graph is of type $(^2A_2)$ or $(^2A_5)$. Let R be a (unique) cuspidal unipotent representation of $M(q)$, with the character χ. Then χ is rational-valued and we have $m_{\boldsymbol{R}}(\chi) = m_{\boldsymbol{Q}_p}(\chi) = 2$ and $m_{\boldsymbol{Q}_l}(\chi) = 1$ for any prime number $l \neq p$.*

This is stated in [5] as (7.6) without detailed proof. We shall now sketch the proof. Let X_f be as in [5], (1.7). Let l be any prime number $\neq p$. For $i \geq 0$, put $H_c^i(X_f) = H_c^i(X_f, \bar{\boldsymbol{Q}}_l) = H_c^i(X_f, \boldsymbol{Q}_l) \otimes \bar{\boldsymbol{Q}}_l$, where $\bar{\boldsymbol{Q}}_l$ is an algebraic closure of $\boldsymbol{Q}_l$. Then $H_c^i(X_f)$ is a $\bar{\boldsymbol{Q}}_l[M(q)]$-module defined over $\boldsymbol{Q}_l$. Let $F: M \to M$ be the Frobenius map. Then F^2 acts on $H_c^i(X_f)$. Let r be the semisimple rank of M. Let V be the F^2-eigensubspace of $H_c^r(X_f)$ corresponding to the eigenvalue $-q$ (resp. $-q^3$) if $r = 2$ (resp. if $r = 5$). Then V is an irreducible $M(q)$-module and is isomorphic to R. As $H_c^r(X_f)$ is defined over $\boldsymbol{Q}_l$ and $\langle R, H_c^r(X_f)\rangle_{M(q)} = 1$, we have $m_{\boldsymbol{Q}_l}(\chi) = 1$. Since $\langle H_c^i(X_f), H_c^j(X_f)\rangle_{M(q)} = 0$ if $i \neq j$, the character of the virtual module $W = \sum(-1)^i H_c^i(X_f)$ is rational-valued and each irreducible component of W has a different degree, χ is rational-valued (see below). By [5], (4.4), there is a $M(q)$-equivariant antisymmetric bilinear form on V. As $\bar{\boldsymbol{Q}}_l \simeq \boldsymbol{C}$, V may be

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