66. An Example of Elliptic Curve over Q with Rank ≥ 20

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Abstract: We construct an elliptic curve over Q with rank ≥ 20.

Mestre [1] (resp. [2]) constructed elliptic curves over $Q(T)$ with $Q(T)$-rank ≥ 11 (resp. with $Q(T)$-rank ≥ 12). In the families of elliptic curves over Q, which are obtained by specialization of above curves, Mestre [3] found an elliptic curve over Q with Q-rank ≥ 15. In choosing appropriate elliptic curves in these families, author [4] (resp. Tunnel (cf. [5]), resp. Fermiger [5]) found two elliptic curves with Q-rank ≥ 17 (resp. one curve with Q-rank ≥ 18, resp. two curves with Q-rank ≥ 19). In this paper, we show by the same method but using a computational device mentioned later that there is an elliptic curve over Q with Q-rank ≥ 20.

§1. Mestre's construction of elliptic curve over $Q(T)$ with $Q(T)$-rank $\geq$ 11. Let $\alpha_i \in Z$ $(i = 1,2,3,4,5,6)$, and put $q(X) = \Pi_{i=1}^{6} (X - \alpha_i)$, $p(X) = q(X - T) * q(X + T) \in Q(T)[X]$. Then there are $g(x)$, $r(X) \in Q(T)[X]$ with deg $g = 6$, deg $r \leq 5$ such that $p = g^2 - r$. Then the curve $Y^2 = r(X)$ contains 12 $Q(T)$-rational points $P_1, \ldots, P_{12}$ where

$$P_i = (T + \alpha_i, g(T + \alpha_i)),\ P_{i+6} = (-T + \alpha_i, g(-T + \alpha_i)),\ {}_{1 \leq i \leq 6}.$$

Let c_5 be the coefficient of X^5 of $r(X)$.

In case $(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = (-17, -16, 10, 11, 14, 17)$, we have $c_5 = 0$ and on the elliptic curve $Y^2 = r(X)$, $P_1, \ldots, P_{11}$ are independent $Q(T)$-rational points. (Group structure is given with P_{12} at origin.)

For any 6-ple of $A = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) \in Z^6$ with $c_5 = 0$, we obtain as above an elliptic curve $\varepsilon_A : Y^2 = r(X)$ over $Q(T)$. For $t \in Q$, we denote with $E_t = E_{A,t}$ the elliptic curve over Q obtained from ε_A by specialization $T \to t$.

§2. Construction of our curve. For an elliptic curve E over Q, and a prime number p, we put $a_p = a_p(E) = p + 1 - \# E(F_p)$. For a fixed integer N, we put furthermore $S(N) = S(N, E) = \sum(-a_p + 2)/(p + 1 - a_p)$ and $S'(N) = S'(N, E) = (\sum - a_p * log(p))/N$ where p runs over good primes satisfying $p \leq N$. It is experimentally known (cf. [6]) that high rank curves are found among curves with large $S(N)$, $S'(N)$.

Now let $A = (95,71,66,58,13,0)$. Then we have $c_5 = 0$. We search in the family of curves

$\{E_{t_1/t_2} (= E_{A,t_1/t_2}) \mid 1 \leq t_1 \leq 3000,\ 1 \leq t_2 \leq 300,\ t_1\ t_2$ *are co-prime*$\}$,

curves satisfying

$S(401) \geq 31.5$, $S'(401) \geq 11$, $S(1987) \geq 61$, $S'(1987) \geq 16$,

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