65. *On a Conjecture on Pythagorean Numbers. II*

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In [1] we considered the following diophantine equation on $l, m, n \in \boldsymbol{N}$

(1) $$(4a^2 - y^2)^l + (4ay)^m = (4a^2 + y^2)^n$$

where $a, y \in \boldsymbol{N}$, with $(a, y) = 1$, $2a > y$, $y \equiv 3 \pmod 4$. l is easily seen to be even. If a is odd, then $m \neq 1 \Leftrightarrow n$ is even. If a is even, then both m and n are even. (Cf. [1] Props. 1-3.) In this paper we consider the case $y = 3$.

Theorem 1. *Let a be even, $a = 2^s a_0$, $(s \geq 1)$, $(2, a_0) = 1$. If the diophantine equation on f, g*

(2) $$4a^2 + 9 = (2^{s+1}f)^2 + (3g)^2$$

has the unique solution $f = a_0$, $g = 1$, then $(l, m, n) = (2, 2, 2)$.

Remark. All even a, with $(a, 3) = 1$, $2 \leq a \leq 152$, except 14, 46, 52, 62, 118, 142, 148, satisfy the above condition.

Proof. As l, m, n are even, put $l = 2l'$, $m = 2m'$, $n = 2n'$ and $(4a^2 + 9)^{n'} + (4a^2 - 9)^{l'} = A$, $(4a^2 + 9)^{n'} - (4a^2 - 9)^{l'} = B$. Then it is proved in [1] that the possibility on choice of A, B in

(3) $$2^{2m}3^m a^m = AB$$

is only the following:

$$A = 2^{m(2+s)-1}b^m, \quad B = 2 \cdot 3^m c^m,$$

where $a_0 = bc$, $(b, c) = 1$, hence l' is odd. (Cf. [1].) If n' is even, then from $A = 2^{m(2+s)-1}b^m$, we have $8 \equiv 0 \pmod{16}$, which is a contradiction. Thus n' is odd, too.

$$(A + B)/2 = (4a^2 + 9)^{n'} = (2^{m'(2+s)-1}b^{m'})^2 + (3^{m'}c^{m'})^2.$$

So

(4) $$(4a^2 + 9)^{n'} = (2^{m'(2+s)-1}b^{m'} + 3^{m'}c^{m'}i)(2^{m'(2+s)-1}b^{m'} - 3^{m'}c^{m'}i).$$

Put $F = 2^{m'(2+s)-1}b^{m'} + 3^{m'}c^{m'}i$, $G = 2^{m'(2+s)-1}b^{m'} - 3^{m'}c^{m'}i$. Then $1 = (F, G)$, as $(b, c) = (b, 6) = (c, 6) = 1$. Therefore there exist integers f_0, g_0 such that $(f_0, g_0) = 1$, $F = (f_0 + g_0 i)^{n'}$, hence $4a^2 + 9 = f_0^2 + g_0^2$. By Lemma 1, which we prove below, we have $3 \mid g_0$, $2^{m'(2+s)-1} \| f_0$, so $2^{s+1} \mid f_0$. By the assumption we have $f_0 = 2a$, $g_0 = 3$. Since $2^{m'(2+s)-1} \| 2a$, $m'(2 + s) - 1 = s + 1$. Thus $m' = 1$, so $m = 2$. Then $A = (4a^2 + 9)^{n'} + (4a^2 - 9)^{l'} = 2^{2(2+s)-1}b^2 \leq 2^{2(2+s)-1}a_0^2 = 8a^2 = (4a^2 + 9) + (4a^2 - 9)$. Therefore $n' = l' = 1$. Thus $(l, m, n) = (2, 2, 2)$.

Lemma 1. *Let a be even and a_0, s, b, c, m', n', F, G as above. If integers f, g with $(f, g) = 1$ satisfy $4a^2 + 9 = f^2 + g^2$ and $2^{m'(2+s)-1}b^{m'} + 3^{m'}c^{m'}i = (f + gi)^{n'}$, then $2^{m'(2+s)-1} \| f$, $3 \mid g$.*

Proof.

$$(f + gi)^{n'} = \sum_{j=0}^{(n'-1)/2} \binom{n'}{2j} f^{n'-2j}(-1)^j g^{2j} + ig \sum_{j=0}^{(n'-1)/2} \binom{n'}{2j+1} f^{n'-(2j+1)}(-1)^j g^{2j}.$$