64. On the Asymptotic Formula for the Number of Representations of Numbers as the Sum of a Prime and a k-th Power

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§1. For an integer $k \geq 2$, let $E_k(X)$ be the number of natural numbers $n \leq X$ such that n is not representable as the sum of a prime and a k-th power. In 1937, Davenport and Heilbronn [3] proved that $E_k(X) = O(X(\log X)^{-c_k})$ with a positive constant c_k depending only on k, in other words, almost all natural numbers are representable as the sum of a prime and a k-th power. After their result, some articles established sharper bounds for $E_k(X)$, and, at present, the best result is $E_k(X) = O(X^{1-\delta_k})$ with a positive constant δ_k depending only on k, which was proved by A. I. Vinogradov [9] and Brünner, Perelli, and Pintz [1] for $k = 2$, and by Plaksin [7] and Zaccagnini [10] for $k \geq 3$. On the difference of the situations between the cases $k = 2$ and $k \geq 3$, we relate in §4 briefly.

On the other hand, let $R_k(n)$ be the number of representations of n as the sum of a prime and a k-th power, $\rho_n(d) = \rho_{n,k}(d)$ be the number of solutions m of the congruence $m^k - n \equiv 0 \pmod d$ with $1 \leq m \leq d$, and let I_k be the set of all natural numbers n such that the polynomial $x^k - n$ is irreducible in $\boldsymbol{Q}[x]$, where $\boldsymbol{Q}$ is the rational number field. As for the asymptotic behavior of $R_k(n)$, it is conjectured that

$$R_k(n) \sim \mathfrak{S}_k(n) \frac{n^{1/k}}{\log n},$$

as n tends to the infinity, providing $n \in I_k$, where

$$\mathfrak{S}_k(n) = \prod_p \left(1 - \frac{\rho_n(p) - 1}{p - 1}\right),$$

and hereafter the letter p stands for prime numbers. For $k = 2$, this was conjectured by Hardy and Littlewood [4, Conjecture H], and Miech [6] proved that

$$R_2(n) = \mathfrak{S}_2(n) \frac{\sqrt{n}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right)$$

for all but $O(X(\log X)^{-A})$ natural numbers $n \leq X$ with any fixed $A > 0$. For each $k \geq 3$, we can also establish an asymptotic formula for $R_k(n)$ valid for almost all n:

Theorem. *For a fixed integer $k \geq 3$, and for any fixed $A > 0$, we have*

$$R_k(n) = \mathfrak{S}_k(n) \frac{n^{1/k}}{\log n} \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) \tag{1}$$

for $n \leq X$ with at most $O(X(\log X)^{-A})$ exceptions.

Because of the possible existence of the Siegel zeros, Miech's result and our result seem the best possible for the present. The proof of our Theorem