48. *A Class of Norms on the Spaces of Schwarzian Derivatives and its Applications**)

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§0. Introduction. As is well-known, the hyperbolic-sup norm (or the Nehari norm) of the Schwarzian derivative of a meromorphic function is closely related to its (global or local) univalence. The famous Nehari-Kraus theorem and Ahlfors-Weill theorem are of fundamental importance in this direction of research.

In this note, in order to clarify this relationship more, we shall introduce, in section 2, a class of "local" norms on the space of Schwarzians. These norms are expected to be near the hyperbolic-sup norm, and determined by the local shape of the domain. But, whereas the pullback by a conformal map is an isometry with the hyperbolic-sup norm, it is only a quasi-isometry with these local norms. In section 3, we shall describe how the magnitude of norms of Schwarzian is controled by the local quasiconformal(= qc) extensibility, which the author has learned from [1]. An essential use of the result in this section will be made in the article [5] of the author. Finally, in section 4, we shall mention an estimate of the local norms of Schwarzian by the injectivity radius.

§1. Preliminaries. Throughout this note, let D be a plane domain of hyperbolic type (i.e., $C\setminus D$ contains at least two points) and $\rho_D(z)\,|\,dz\,|$ be the hyperbolic metric with constant negative curvature -4. For a holomorphic function φ on D, we define the hyperbolic-sup norm of φ by $\|\varphi\|_D = \sup_{z\in D}\rho_D(z)^{-2}\,|\,\varphi(z)\,|$ and we denote by $B_2(D)$ the space of all holomorphic functions in D with a finite norm, which becomes a complex Banach space. For a non-constant meromorphic function f on D, the Schwarzian derivative of f is defined by the formula $S_f = (f''/f')' - \frac{1}{2}(f''/f')^2$, which is holomorphic at $z_0 \in D$ if and only if f is locally univalent at z_0.

In this note, $f:\hat{C}\to\hat{C}$ shall be called a *k-qc map* of $\hat{C}$ where k is a constant and $0\le k<1$, if f is an orientation-preserving self-homeomorphism of the Riemann sphere $\hat{C}$ with locally L^2-derivatives such that $|\partial_{\bar z}f|\le k\,|\partial_z f|$ a.e. It should be alerted that this terminology is not standard. In fact, k-qc map is ordinarily called "K-qc" where $K=\dfrac{1+k}{1-k}$. As a general reference for qc maps and the hyperbolic sup-norm of the Schwarzian derivatives, we refer to [4].

*) Dedicated to Professor Nobuyuki Suita on his sixtieth birthday.